Relativistic Larmor's formula

non relativistic result was

$$P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q}{c^3}a^2$$

total power radiated by particle with acceleration $\vec{a}$ assuming $v \ll c$

Now consider a particle moving with any speed v.
Consider the inertial frame of reference in which that particle is instantaneously at rest. Call this frame $\overset{\circ}{K}$. The velocity in this frame is thus $\overset{\circ}{\vec{v}} = 0$, and the charge is at the origin ~~....~~ $\overset{\circ}{\vec{r}} = 0$.
The power radiated, as seen in the frame $\overset{\circ}{K}$, is then exactly

$$\overset{\circ}{P} = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q}{c^3}\overset{\circ}{a}^2$$

where $\overset{\circ}{a}$ is the acceleration in frame $\overset{\circ}{K}$.

This result is exact because as $v/c \to 0$ all terms higher than the electric dipole term will vanish.

What we need to do is to find the way to Lorentz transform the result $\overset{\circ}{P}$ and find its value in any other frame of reference, in which the particle is moving with any velocity $\vec{v}$.

Consider the momentum-energy 4-vector describing the total momentum and total energy of the electromagnetic fields ~~of the charge~~ radiated by the charge.

in frame $\overset{\circ}{K}$ we can write this as

$$\left(\vec{\overset{\circ}{P}}_{EM}, \ \frac{i\overset{\circ}{\mathcal{E}}}{c}\right)$$

Now $\vec{\overset{\circ}{P}}_{EM} = \int d^3\overset{\circ}{r} \ \varepsilon_0 \vec{\overset{\circ}{E}} \times \vec{\overset{\circ}{B}}$.

But since the radiated fields are in the radial direction $\hat{\overset{\circ}{r}}$, when we integrate over all space we find $\vec{\overset{\circ}{P}}_{EM} = 0$.

Alternatively you have from homework, for a charge moving with small velocity $\vec{v}$, $\vec{P}_{EM} = \frac{4}{3}\frac{U}{c^2}\vec{v}$
So when $\vec{v} \to 0$, $\vec{P}_{EM} \to 0$.

So in frame $\overset{\circ}{K}$ the momentum energy 4-vector is

$$\left(0, \ \frac{i\overset{\circ}{\mathcal{E}}}{c}\right)$$

In a new frame of reference K that moves with velocity $-\vec{v}$ with respect to $\overset{\circ}{K}$ (in frame K, the charge is moving with velocity $\vec{v}$) the energy in frame K is obtained by the transformation law for 4-vectors

$$\frac{i\mathcal{E}}{c} = \gamma\left(\frac{i\overset{\circ}{\mathcal{E}}}{c} + \frac{i v}{c}\overset{\circ}{P}_{EM\,1}\right)$$

where $\overset{\circ}{P}_{EM\,1}$ is component of $\vec{\overset{\circ}{P}}_{EM}$ in direction of $\vec{v}$. But $\vec{\overset{\circ}{P}}_{EM} = 0$

So $\quad i\frac{\mathcal{E}}{c} = \gamma i \frac{\mathring{\mathcal{E}}}{c} \Rightarrow \mathcal{E} = \gamma \mathring{\mathcal{E}}$

similarly, if we take the origins of K and $\mathring{K}$ to coincide at the time when we are measuring the radiated power, then time transforms as

$$t = \gamma \mathring{t} + \frac{v}{c^2}\gamma \mathring{x}_1$$

where $\mathring{x}_1$ is position of charge in direction of $\vec{v}$

But charge is at origin in $\mathring{K}$ so $\mathring{x}_1 = 0$

So $\quad t = \gamma \mathring{t}$ ⟵ (since charge is not moving in $\mathring{K}$, $d\mathring{t}$ is really the proper time ds, so this is the familiar $\frac{dt}{\gamma} = ds$)

The Power radiated in frame K is then

$$\mathcal{P} = \frac{d\mathcal{E}}{dt} = \frac{\gamma\, d\mathring{\mathcal{E}}}{\gamma\, d\mathring{t}}$$

transforming $\mathcal{E} = \gamma \mathring{\mathcal{E}}$, $t = \gamma \mathring{t}$

$$= \frac{d\mathring{\mathcal{E}}}{d\mathring{t}} = \mathring{\mathcal{P}}$$

So the total radiated power is a Lorentz invariant scalar!

$$\boxed{\mathcal{P} = \mathring{\mathcal{P}} = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}q\frac{\mathring{a}^2}{c^3}}$$

where $\mathring{a}$ is acceleration of charge in its rest frame

We would like to rewrite $\mathcal{P}$ in a way that makes no explicit reference to the frame $\mathring{K}$.
ie we want to write $\mathring{a}^2$ in terms of a Lorentz invariant scalar that may be evaluated in any frame K.

Consider the 4-acceleration

$$\alpha_\mu = \frac{d u_\mu}{ds} = \gamma \frac{d}{dt} u_\mu \qquad \text{since } ds = dt/\gamma$$

use $u_\mu = (\gamma \vec{v}, i\gamma c)$

$$\vec{\alpha} = \gamma \frac{d}{dt}(\gamma \vec{v}) = \gamma^2 \vec{a} + \gamma \vec{v} \frac{d\gamma}{dt}$$

$$\alpha_4 = \gamma i c \frac{d\gamma}{dt}$$

we need $\dfrac{d\gamma}{dt} = \dfrac{d}{dt}\left(\dfrac{1}{\sqrt{1-v^2/c^2}}\right) = \dfrac{+\frac{\vec{v}}{c^2} \cdot \frac{d\vec{v}}{dt}}{(1-v^2/c^2)^{3/2}}$

$$= + \frac{\vec{v} \cdot \vec{a}}{c^2} \gamma^3$$

so

$$\vec{\alpha} = \gamma^2 \vec{a} + \gamma^4 \left(\frac{\vec{v} \cdot \vec{a}}{c^2}\right) \vec{v}$$

$$\alpha_4 = \gamma^4 i \left(\frac{\vec{v} \cdot \vec{a}}{c}\right)$$

$$\alpha_\mu = \gamma^4 \left(\left(\frac{\vec{v} \cdot \vec{a}}{c^2}\right) \vec{v} + \frac{\vec{a}}{\gamma^2}, \; i \left(\frac{\vec{v} \cdot \vec{a}}{c}\right) \right)$$

in frame $\overset{\circ}{K}$, $\overset{\circ}{\vec{v}} = 0$ and $\overset{\circ}{\gamma} = 1$, so

$$\overset{\circ}{\alpha}_\mu = (\overset{\circ}{\vec{a}}, 0) \quad \text{and so} \quad \overset{\circ}{a}^2 = \alpha_\mu^2 \quad \text{Lorentz invariant scalar}$$

So now we can write the relativistic Larmor formula

$$P = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q}{c^3} \overset{\circ}{a}^2 = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q}{c^3} \alpha_\mu^2 \quad \text{in any frame } K$$

In a general frame K,

$$\alpha_\mu^2 = |\vec{\alpha}|^2 + \alpha_4^2$$

$$= \gamma^8 \left[\left(\frac{\vec{v}\cdot\vec{a}}{c^2}\right)^2 v^2 + \frac{a^2}{\gamma^4} + 2\left(\frac{\vec{v}\cdot\vec{a}}{c^2}\right)\frac{(\vec{v}\cdot\vec{a})}{\gamma^2} - \left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2 \right]$$

$$= \gamma^8 \left[-\left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2 \left(1 - \frac{v^2}{c^2}\right) + 2\left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2 \frac{1}{\gamma^2} + \frac{a^2}{\gamma^4} \right]$$

$$= \gamma^8 \left[\left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2 \left(\frac{2}{\gamma^2} - \frac{1}{\gamma^2}\right) + \frac{a^2}{\gamma^4} \right]$$

$$= \gamma^8 \left[\frac{a^2}{\gamma^4} + \left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2 \frac{1}{\gamma^2} \right]$$

$$\alpha_\mu^2 = \gamma^4 \left[a^2 + \gamma^2 \left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2 \right]$$

Note: as $v \to 0$, $\gamma \to 1$ and we get $\alpha_\mu^2 = a^2$ as we must.

So power radiated is

$$P = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q}{c^3} \gamma^4 \left[a^2 + \gamma^2 \left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2 \right]$$

<u>Examples</u>:

(1) For a charge accelerating in linear motion (such as in a linear particle accelerator such as SLAC)

$\vec{v}\cdot\vec{a} = va$ since $\vec{v}$ and $\vec{a}$ are colinear

$$P = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q}{c^3} \gamma^4 \left[a^2 + \gamma^2 \frac{v^2 a^2}{c^2} \right]$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q}{c^3} \gamma^4 a^2 \left[1 + \gamma^2 \frac{v^2}{c^2} \right]$$

$$1 + \gamma^2 \frac{v^2}{c^2} = 1 + \frac{v^2/c^2}{1 - v^2/c^2} = \frac{1}{1 - v^2/c^2} = \gamma^2$$

$$P = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q}{c^3} a^2 \gamma^6$$

relativistic result increased by factor γ^6 compared to non-relativistic result

② For a charge accelerating in circular motion (such as in a synchrotron)

$\vec{v} \cdot \vec{a} = 0$ since $\vec{v} \perp \vec{a}$

$$P = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q}{c^3} a^2 \gamma^4$$

relativistic result increased by factor γ^4 compared to non-relativistic result