Magnetic Dipole moment in magnetostatics

In the Coulomb gauge we have

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

we can expand $|\vec{r}-\vec{r}'|$ assuming that we want $\vec{A}$ at $\vec{r}$ that is far from a localized source at $\vec{r}'$.

$$|\vec{r}-\vec{r}'| = \sqrt{r^2 + r'^2 - 2\vec{r}\cdot\vec{r}'} = r\sqrt{1 + \left(\frac{r'}{r}\right)^2 - 2\frac{\hat{r}\cdot\vec{r}'}{r}}$$

$$\simeq r\left(1 - \frac{\hat{r}\cdot\vec{r}'}{r}\right)$$ we we expand $\sqrt{1+\delta} \approx 1 + \frac{\delta}{2}$ to lowest order in (r'/r)

$$\frac{1}{|\vec{r}-\vec{r}'|} \simeq \frac{1}{r}\left(1 + \frac{\hat{r}\cdot\vec{r}'}{r}\right)$$ expanding $\frac{1}{1+\delta} \sim 1 - \delta$

$$\vec{A}(\vec{r}) \simeq \frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r' \, \vec{j}(\vec{r}')\left[1 + \frac{\hat{r}\cdot\vec{r}'}{r}\right]$$

$$= \frac{\mu_0}{4\pi}\frac{1}{r}\left[\vec{I}_1 + \frac{1}{r}\vec{I}_2\right]$$

with $\vec{I}_1 = \int d^3r' \, \vec{j}(\vec{r}')$, $\vec{I}_2 = \int d^3r \, (\hat{r}\cdot\vec{r}')\vec{j}(\vec{r}')$

These are just the same $\vec{I}_1$ and $\vec{I}_2$ we saw in our discussion of radiation, except here they are for magnetostatic currents $\vec{j}(\vec{r})$ such that $\frac{\partial \vec{j}}{\partial t} = 0$ and $\vec{\nabla}\cdot\vec{j} = 0$

We can apply the calculations that we did in our discussion of radiation, provided we take $\omega = 0$, since $\vec{j}$ is constant in time.

There we had $\vec{I}_1 = -i\omega \vec{p}_\omega = 0$ when $\omega = 0$
so this term vanishes $\Rightarrow$ in expansion for $\vec{A}$ there is no $1/r$ term. This is equivalent to saying there is no magnetic monopole term in magnetostatics

$$\vec{I}_2 = -\hat{r} \times \vec{m} - \frac{1}{2} \frac{i\omega}{3} \hat{r} \cdot \overleftrightarrow{Q}'$$
$$= -\hat{r} \times \vec{m}$$
second term vanishes when $\omega = 0$

where $\boxed{\vec{m} \equiv \frac{1}{2} \int d^3r \; \vec{r} \times \vec{j}(\vec{r})}$

So to this order of approximation

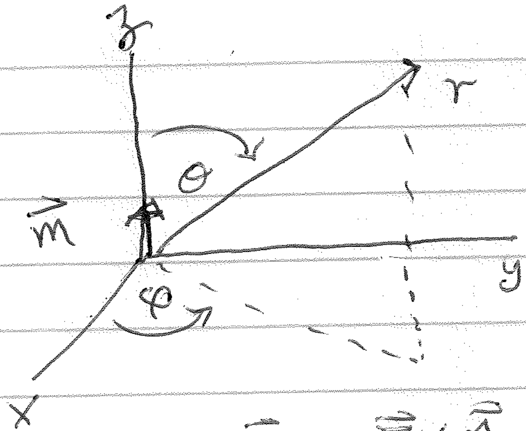
$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{I}_2}{r^2} = -\frac{\mu_0}{4\pi} \frac{\hat{r} \times \vec{m}}{r^2} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$$

This is the magnetic dipole approximation $\vec{A} \sim \frac{1}{r^2}$

The magnetic field, in the magnetic dipole approx, is given by

$$\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \vec{\nabla} \times \left(\frac{\vec{m} \times \hat{r}}{r^2} \right)$$

If we choose $\vec{m}$ to lie along the z-axis
$\vec{m} = m\hat{z}$ then $\vec{m} \times \hat{r} = m \sin\theta \, \hat{\varphi}$



$$\vec{m} \times \hat{r} = |\vec{m}||\hat{r}| \sin\theta \; \hat{\varphi}$$
$$= m \sin\theta \; \hat{\varphi}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{m \sin\theta}{r^2} \hat{\varphi}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial\theta} (\sin\theta \, A_\varphi) \right] \hat{r}$$
$$- \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\varphi) \right] \hat{\theta}$$

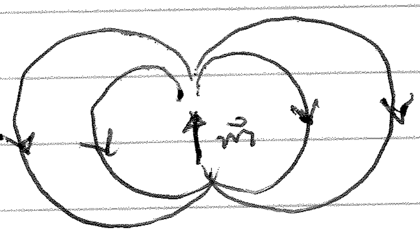
in spherical coords, with $A_r = A_\theta = 0$

$$\vec{B} = \frac{\mu_0 m}{4\pi r^3} \left[2\cos\theta \, \hat{r} + \sin\theta \, \hat{\theta} \right]$$

same form as $\vec{E}$ from an electric dipole

or

$$\vec{B} = \frac{\mu_0}{4\pi r^3} \left[3(\vec{m} \cdot \hat{r}) \hat{r} - \vec{m} \right]$$



field lines of $\vec{B}$ from a magnetic dipole

to show last two expressions are equal, use

$$\vec{m} \cdot \hat{r} = m \cos\theta$$
$$\vec{m} = (\vec{m} \cdot \hat{r}) \hat{r} + (\vec{m} \cdot \hat{\theta}) \hat{\theta} = m \cos\theta \, \hat{r} + m \sin\theta \, \hat{\theta}$$

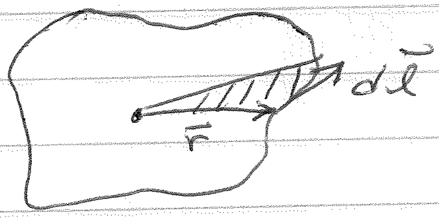
So $\quad 3(\vec{m} \cdot \hat{r}) \hat{r} - \vec{m} = 3m\cos\theta \, \hat{r} - m\cos\theta \, \hat{r} - m \sin\theta \, \hat{\theta}$

$$= 2m \cos\theta \, \hat{r} - m \sin\theta \, \hat{\theta}$$

For a current carrying loop in a plane

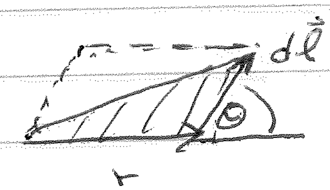
$$\vec{m} = \frac{1}{2}\int d^3r\ \vec{r}\times\vec{J} = \frac{1}{2} I \oint \vec{r}\times d\vec{l}$$

since $d^3r\ \vec{r}\times\vec{J} = \vec{r}\times I d\vec{l}$



$\vec{r}\times d\vec{l} = r\,dl \sin\theta\ \hat{n}$

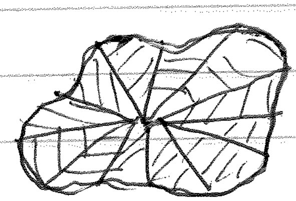
normal to plane of loop use right hand rule for direction



shaded area is $(r\,dl \sin\theta)(\frac{1}{2})$

$\frac{1}{2}$ area of the parallelogram = $\frac{1}{2}$(base)(height)

$$\vec{m} = \hat{n} I \oint dl \frac{r\sin\theta}{2} = \hat{n}\, I\,(\text{area of loop})$$



each wedge has area $\frac{r\,dl\sin\theta}{2}$

so

$$\vec{m} = I(\text{area})\,\hat{n}$$

normal to plane of loop