Statics Review

charge density $\rho(\vec{r})$

$$\int_{\text{volume } V} d^3r\, \rho(\vec{r}) = Q \quad \text{total charge inside volume } V$$

current density $\vec{j}(\vec{r})$

$$\int_{\text{surface } S} d\vec{a} \cdot \vec{j}(\vec{r}) = I \quad \text{total current}$$

(charge per unit time) flowing through surface S

local charge conservation

$$\frac{\partial \rho(\vec{r})}{\partial t} + \vec{\nabla} \cdot \vec{j}(\vec{r}) = 0$$

in electrostatics, $\frac{\partial \rho}{\partial t} = 0 \Rightarrow \vec{\nabla} \cdot \vec{j}(\vec{r}) = 0$ is key condition for magnetostatics

– also $\frac{\partial \vec{j}}{\partial t} = 0$

Maxwell's Equations

	electrostatics	magnetostatics
integral form:	Gauss $\oint_{\text{surface } S} \vec{E} \cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_0}$	$\oint_S \vec{B} \cdot d\vec{a} = 0$
	$\oint_{\text{curve } C} \vec{E} \cdot d\vec{l} = 0$	Ampere $\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$
differential form:	$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_0}$	$\vec{\nabla} \cdot \vec{B}(\vec{r}) = 0$
	$\vec{\nabla} \times \vec{E}(\vec{r}) = 0$	$\vec{\nabla} \times \vec{B}(\vec{r}) = \mu_0 \vec{j}(\vec{r})$

Potentials

electrostatic potential $V(\vec{r})$

$$\vec{E} = -\vec{\nabla} V$$

magnetostatic vector potential $\vec{A}(\vec{r})$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

can also choose $\vec{A}$ to satisfy the additional condition $\vec{\nabla} \cdot \vec{A} = 0$ see Griffiths sec ~~[illegible]~~ 5.4.1

"Coulomb Gauge"

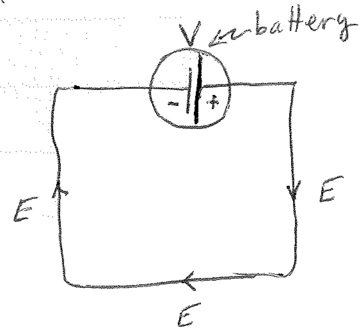
~~Read 6.4.2 ferromag~~

Magnetic Induction + Faradays Law   Read 7.1.1

electromotive force ≡ "emf" ≡ $\mathcal{E}$ is the work done, per unit charge, to move a current around a closed loop

$$\mathcal{E} = \oint \vec{f}\cdot d\vec{l} \quad \text{where } \vec{f} \text{ is the force per unit charge.}$$

ex: For a single wire loop of length L, connected to a battery of voltage V, there is an electric field in the wire $E = \frac{V}{L}$, pointing tangential to the wire



force on electrons is $\vec{F} = -e\vec{E}$

force per unit charge is $\vec{f} = \frac{\vec{F}}{-e} = \vec{E}$

work done per electron is

$$\mathcal{E} = \oint \vec{E}\cdot d\vec{l} = EL = V$$

in doing integral, we left out the section containing the battery. If we included the section containing the battery, then in this portion $\int \vec{E}\cdot d\vec{l} = -V$, since $\oint E\cdot dl$ around a closed loop is zero as $\vec{\nabla}\times\vec{E} = 0$; but we would also have to include the chemical force supplied by the battery, and this $\int \vec{f}_{battery}\cdot d\vec{l} = V$ - this just defines the voltage supplied by the battery. So the electric and chemical force contributions cancell in the battery segment, and the net emf is $\mathcal{E} = V$ as computed above.

In definition of $\mathcal{E}$, direction one goes around loop is arbitrary.
If $\mathcal{E}$ comes out > 0, then current flows in direction of integration
If $\mathcal{E}$ comes out < 0, then current flows in opposite direction to integration

Induction

A changing magnetic flux through a loop, induces an emf around the loop

$$\boxed{\mathcal{E} = -\frac{d\Phi}{dt}}$$

where $\Phi = \int \vec{B}\cdot d\vec{a}$ flux of $\vec{B}$ through the loop

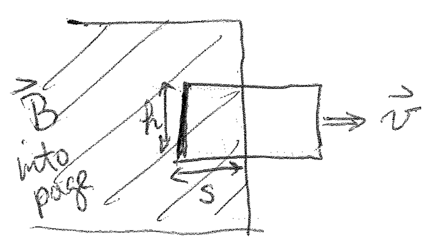
integrate over area bounded by loop

signs: if go ↺ in computing $\mathcal{E}$, then take outward normal $d\vec{a} = \hat{n}\,da$ in computing Φ.

if go ↻ in computing $\mathcal{E}$, then take inward normal in computing Φ.

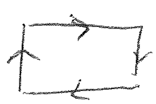
Always use right hand rule.

proof in a simple example



pull wire loop as shown

as pull loop, charges in left side experience Lorentz force $\vec{F} = q\,\vec{v}\times\vec{B}$ causing current to flow upwards along wire (Lorentz forces on top + bottom segments are $\perp$ to wire, so they dont drive any current)



integrate clockwise

$$\mathcal{E} = \oint \vec{f}_{mag}\cdot d\vec{l} = \oint (\vec{v}\times\vec{B})\cdot d\vec{l} = vBh$$

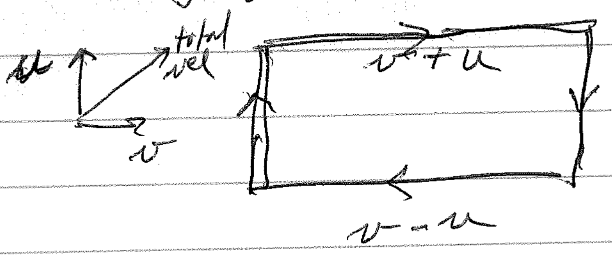
flux through loop 5 $\Phi = B h s$

$$\frac{d\Phi}{dt} = B h \frac{ds}{dt} = -B h v$$

sign negative as s is decreasing

$$\Rightarrow \mathcal{E} = -\frac{d\Phi}{dt}$$

When we worked out the emf around the loop above we considered the velocity $\vec{v}$ of the loop, but we ignored the velocity of the charges going around the loop. Lets now add that, say the speed ~~velocity~~ of charges traveling down the wire is u.



$\Longrightarrow$ loop pulled to right with $\vec{v}$

The Lorentz force on the horizontal segments of the wire is still in the vertical direction $\Rightarrow$ does not contribute to the emf.

The Lorentz force on the left vertical segment is

$$q(\vec{v}+\vec{u}) \times \vec{B} = q\vec{v} \times \vec{B} + q\vec{u} \times \vec{B}$$

oriented parallel to the wire — contributes to emf

oriented opposite to direction $\vec{v}$, or orthogonal to wire — does not contribute to emf

So the velocity $\bar{u}$ of the charges flowing down the wire does <u>not</u> contribute to the emf around the loop.

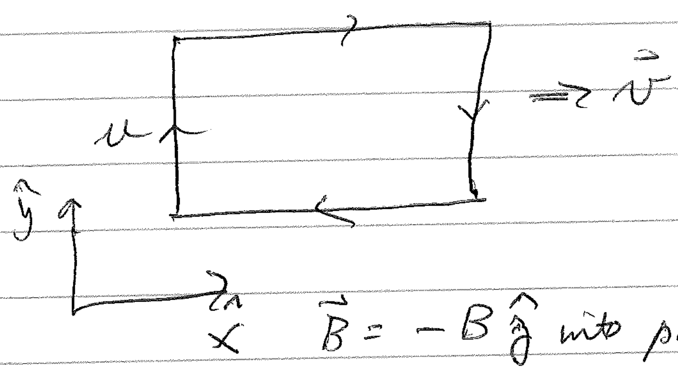
But $\bar{u}$ is important for something else!

What is the source of the work that drives the current around the loop?
It is not $\vec{B}$ since magnetic fields can do no work!

$$\vec{F}_L = q\bar{v} \times \bar{B} \text{ is } \perp \vec{v} \text{ so } \bar{v} \cdot \vec{F}_L = 0$$

↑ work per time done by $\vec{F}_L$ on particle

<u>Answer</u>: It is the person pulling the loop that is doing the work



$\vec{B} = -B\hat{z}$ into page

As we computed above, the motion of the charges in the left vertical segment of the loop gives rise to a Lorentz force

$$\vec{F}_L = q\,\bar{u} \times \bar{B} = q(u\hat{y}) \times (-B\hat{z})$$
$$= -quB\hat{x}$$

~~The trip down this left segment of the loop takes~~

So the person pulling the loop must exert a force equal and opposite to this, per charge, to pull the loop and keep it moving with constant $\vec{v}$.

$F_{pull} = quB\hat{x}$ ~~Work done by puller, per charge for one trip of charge around loop is~~

Work is only done as the charge moves down left segment, since in other ~~segm~~ horizontal segments $\vec{F}_L$ is in $\hat{y}$ direction and does not oppose force of puller.

Time for charge to flow down left segment is

$$\Delta t = \frac{h}{u}$$

During Δt the loop moves $\Delta s = \Delta t\, v$

so work done by puller to move the charge around the loop is

$$quB\,\Delta s = quB\,\Delta t\, v = quB\left(\frac{h}{u}\right)v$$

$$= q v B h$$

Work per charge ~~[illegible]~~ is

$$W = vBh = \mathcal{E} = -\frac{d\Phi}{dt}$$

So puller does just the right amount of mechanical work to provide the emf driving the current!

~~the trip down the left segment takes a time~~ $\cancel{\Delta t = \frac{h}{u}}$.
~~In this time, the distance travelled in y direction is~~ $\cancel{v \Delta t = h \frac{v}{u}}$
~~ie:~~ $\cancel{d\vec{l} = h\hat{z} + h\frac{v}{u}\hat{y}}$

$$\cancel{\int u B \hat{y} \cdot d\vec{l} = u B h \frac{v}{u} = B h v = \mathcal{E}}.$$

~~So work done by puller = emf around loop~~

$\mathcal{E} = -\frac{d\Phi}{dt}$ is true in general – need not be spatially uniform B, ~~need not be square loop~~ loop may have any shape, or even may change shape as time varies. see proof in text. pg 307-308

<u>Faraday's Law</u>

In previous example, suppose loop is fixed, but magnet moves to left with velocity v. By special relativity, we expect same result as before, ie an emf is induced in loop

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

Now what force creates the emf? Since loop (& the charges in it) are <u>stationary</u>, they experience <u>no</u> Lorentz force. Faraday $\Rightarrow$ charges feel an electric force!

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt}$$

changing B induces an E in loop.

note: since there is no battery, or anything else in the loop driving the current, $\oint \vec{E}\cdot d\vec{l}$ goes completely around loop. But this is inconsistent with electrostatic $\vec{\nabla}\times\vec{E}=0$.
$\Rightarrow$ $\vec{\nabla}\times\vec{E}\neq 0$ in general, when $\vec{B}$ varies in time.

$$\mathcal{E} = -\frac{d\Phi}{dt} \Rightarrow \oint \vec{E}\cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}$$

$$\overset{\text{Stokes}}{\parallel}$$

$$\int (\vec{\nabla}\times\vec{E})\cdot d\vec{a} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$ Faraday's Law.

When $\frac{\partial \vec{B}}{\partial t} = 0$, ie statics, we regain $\vec{\nabla}\times\vec{E}=0$.

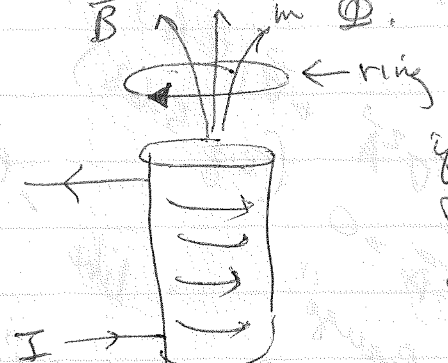
Faraday's law holds true no matter what cause B to change in time, ie same result if magnet moves, or strength of stationary electromagnet is changed by increasing current in solenoid.

Note: If looked at in magnet's rest frame, where loop is moving, the force that moves the charges is magnetic
If looked at in loop's rest frame, where magnet is moving, the force that moves the charges is electric.

$\Rightarrow$ electric forces + magnetic forces depend on the reference frame from which they are viewed
(this is obvious for $F_{mag} = q\vec{v}\times\vec{B}$ as relativity $\Rightarrow$ there is no absolute way to measure $\vec{v}$)

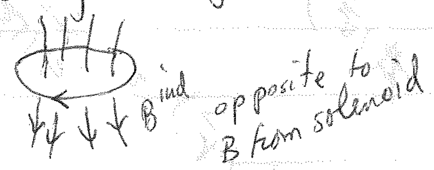
⇒ electric and magnetic fields are part of same physical phenomena. This was one of motivations for Einstein in developing special relativity.

<u>Lenz law</u> $\frac{\partial \Phi}{\partial t}$ induces an emf $\mathcal{E}$ that drives a current in the direction such that the magnetic fields produced by this current, tend to oppose the change in Φ.



if increase current in solenoid, B through ring increases. $\mathcal{E}$ drives current as shown. B_{ind} created by this induced current has negative flux through ring, ie tends to oppose increasing flux from solenoid. Opposite circulating currents in ring + solenoid will repell + ring will jump up!

B field produced by current circulating in ring is



Note: <u>If</u> we have situation where charge density $\rho = 0$, then equ's for $\vec{E}$ are

$$\left.\begin{aligned} \vec{\nabla}\cdot\vec{E} &= 0 \\ \vec{\nabla}\times\vec{E} &= -\frac{\partial \vec{B}}{\partial t} \end{aligned}\right\} \text{same form as magnetostatics !} \qquad \begin{aligned} \vec{\nabla}\cdot\vec{B} &= 0 \\ \vec{\nabla}\times\vec{B} &= \mu_0 \vec{J} \end{aligned}$$

we can solve for $\vec{E}$ using methods from magnetostatics!

$$\Rightarrow \cancel{\vec{E}(\vec{r}) = -\frac{1}{4\pi}\int d^3r' \, \frac{\partial \vec{B}(\vec{r}')}{\partial t} \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}} \text{ ~~is solution~~}$$

$$\cancel{= -\frac{\partial}{\partial t}\left[\frac{1}{4\pi}\int d^3r' \, \vec{B}(\vec{r}') \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right]}$$