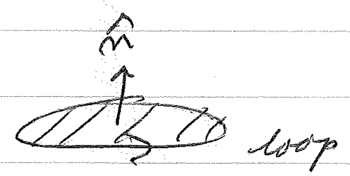
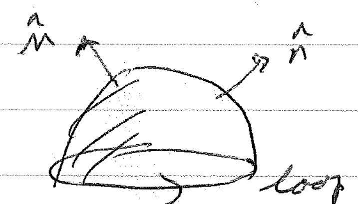
Magnetic Flux

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

When we compute the flux through a loop, does it matter what surface we use to compute Φ?



surface S is flat surface in plane of loop



surface S' is hemisphere bounded by loop.

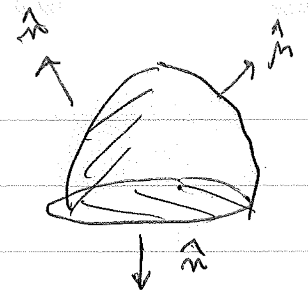
$$\int_S d\vec{a}\cdot\vec{B} \stackrel{?}{=} \int_{S'} d\vec{a}\cdot\vec{B}$$

YES!! all surfaces bounded by the same loop will give the same $\Phi = \int d\vec{a}\cdot\vec{B}$

Follows from fact that $\vec{\nabla}\cdot\vec{B} = 0$. To see this:

method I: Consider the two surfaces S and S' as above. Construct the closed surface $S'' = S' - S$ where $-S$ is the same as S but with $\hat{n}$ in the opposite direction.

S'' is just the hemisphere S' closed off by the planar S at the bottom



$$\oint_{S''} d\vec{a}\cdot\vec{B} \overset{\text{Gauss' Theorem}}{=} \int_V d^3r\, \vec{\nabla}\cdot\vec{B} = 0 \quad \text{since } \vec{\nabla}\cdot\vec{B}=0$$

$$\oint_{S''} d\vec{a}\cdot\vec{B} = \oint_{S'} d\vec{a}\cdot\vec{B} - \oint_{S} d\vec{a}\cdot\vec{B} = 0$$

$$\Rightarrow \oint_{S} d\vec{a}\cdot\vec{B} = \oint_{S''} d\vec{a}\cdot\vec{B}$$ same through both surfaces

method II

Since $\vec{\nabla}\cdot\vec{B}=0$, we can write $\vec{B}=\vec{\nabla}\times\vec{A}$

Then $\Phi = \int_S d\vec{a}\cdot\vec{B} = \int_S d\vec{a}\cdot(\vec{\nabla}\times\vec{A}) = \oint_C d\vec{l}\cdot\vec{A}$

by Stokes theorem, where C is the curve bounding S',

Now $\oint_C d\vec{l}\cdot\vec{A}$ depends only on the curve C, not on the surface S, so Φ does not depend on S - only C.

Does Φ depend on what gauge we used for $\vec{A}$? No!

$\vec{A}' = \vec{A} + \vec{\nabla}\lambda$, λ any scalar function. Then

$$\oint_C d\vec{l}\cdot\vec{A}' = \oint_C d\vec{l}\cdot\vec{A} + \underbrace{\oint_C d\vec{l}\cdot\vec{\nabla}\lambda}_{=0}$$

So $\oint_C d\vec{l}\cdot\vec{A}$ is the same for all $\vec{A}$ that give the same $\vec{B}$

Maxwell's Equations with Faraday's law

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

Suppose $\rho = 0$ no charges, and we know $\frac{\partial\vec{B}}{\partial t}$. Can we solve for $\vec{E}$?

$$\left.\begin{array}{l}\vec{\nabla}\cdot\vec{E} = 0 \\ \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}\end{array}\right\} \text{has the same form as magnetostatics with } \vec{B}\to\vec{E},\ \mu_0\vec{J}\to -\frac{\partial\vec{B}}{\partial t} \left\{\begin{array}{l}\vec{\nabla}\cdot\vec{B} \\ \vec{\nabla}\times\vec{B} = \mu_0\vec{J}\end{array}\right.$$

Provided the source is localized in space, we had the solution for magnetostatics as follows

$$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla}\times\vec{A}$$
$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = -\nabla^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = \mu_0\vec{J}$$

in Coulomb gauge we have $\vec{\nabla}\cdot\vec{A} = 0$, so then $\underline{-\nabla^2\vec{A} = \mu_0\vec{J}}$

Solution is $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$ solution to Poisson's Eqn

to get $\vec{B}$ use $\vec{B} = \vec{\nabla}\times\vec{A}$

To get the magnetic field we now use $\vec{B} = \vec{\nabla} \times \vec{A}$

$$\vec{B}(\vec{r}) = \vec{\nabla} \times \left[\frac{\mu_0}{4\pi} \int d^3r' \; \frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \right]$$

derivatives act on coordinate $\vec{r}$, not $\vec{r}\,'$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \left[\vec{\nabla} \times \left(\frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \right) \right]$$

term in the [ ] has the form $\vec{\nabla} \times (\vec{j} f(\vec{r}))$ where $\vec{j}$ is a constant vector, and $f(\vec{r})$ is a scalar function of position. From the front cover of Griffiths we then have

$$\vec{\nabla} \times (f\vec{j}) = f\underbrace{(\vec{\nabla} \times \vec{j})}_{=0} - \vec{j} \times (\vec{\nabla} f)$$

when $\vec{j}$ is a constant

$$\text{So} \quad \vec{\nabla} \times \left(\frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \right) = -\vec{j}(\vec{r}\,') \times \vec{\nabla} \left(\frac{1}{|\vec{r}-\vec{r}\,'|} \right)$$

$$-\vec{\nabla} \left(\frac{1}{|\vec{r}-\vec{r}\,'|} \right) = \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \quad \text{where } \vec{\mathscr{r}} = \vec{r}-\vec{r}\,'$$

( ) is potential of charge with $q=1$

electric field of charge with $q=1$

$-\vec{\nabla} V = \vec{E}$

$$\text{So} \quad \boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \; \vec{j}(\vec{r}\,') \times \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}}$$

This is just the Biot - Savart Law!

Now apply to our problem of finding $\vec{E}$ when $\rho = 0$ but $\frac{\partial \vec{B}}{\partial t} \neq 0$

Gauss $\vec{\nabla} \cdot \vec{E} = 0$ compare to $\vec{\nabla} \cdot \vec{B} = 0$

Faraday $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ $\vec{\nabla} \cdot \vec{B} = \mu_0 \vec{J}$

solution for $\vec{E}$ obtained from our magnetostatic solution by making the substitutions

$$\vec{B} \to \vec{E}, \quad \mu_0 \vec{J} \to -\frac{\partial \vec{B}}{\partial t}$$

So

$$\boxed{\vec{E}(\vec{r},t) = \frac{-1}{4\pi} \int d^3r' \left(\frac{\partial \vec{B}(\vec{r}',t)}{\partial t}\right) \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}}$$

$$= -\frac{\partial}{\partial t}\left[\frac{1}{4\pi} \int d^3r' \, \vec{B}(r',t) \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right]$$

So if we know $\vec{B}(\vec{r},t)$ we can find the induced $\vec{E}(\vec{r},t)$.

You may have seen the Biot-Savart law just for the case of a current carrying wire, where $\vec{J} \neq 0$ only along the path of a one dimensional curve.

In that case $d^3r' \, \vec{J}(r') = d\vec{l}' \, I$

differential tangent to curve

and so

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int d\vec{l}' \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$ Griffiths Eq (5.34)

Levi - Civita Symbol

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is even permutation of } 123 \\ -1 & \text{if } ijk \text{ is odd permutation of } 123 \\ 0 & \text{otherwise, ie if any two of the } ijk \text{ are equal} \end{cases}$$

ijk is an $\binom{\text{even}}{\text{odd}}$ permutation of 123 if you can get to it from 123 by making an $\binom{\text{even}}{\text{odd}}$ number of pairwise interchanges.

Example: 213 is an odd permutation $\underbrace{123}_{\text{one switch}} \to 213$

231 is an even permutation $\underbrace{123}_{\text{switch}} \to \underbrace{213}_{\text{switch}} \to 231$

If $\vec{A} = \vec{B} \times \vec{C}$ then ith component of $\vec{A}$ is given by

$$A_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} B_j C_k$$

For example, $A_1 = \sum_{j,k} \varepsilon_{1jk} B_j C_k$

all other terms vanish

$$= \varepsilon_{123} B_2 C_3 + \varepsilon_{132} B_3 C_2$$

$$A_1 = B_2 C_3 - B_3 C_2 \quad \text{correct!}$$

Similarly

$$A_2 = \varepsilon_{231} B_3 C_1 + \varepsilon_{213} B_1 C_3$$
$$= B_3 C_1 - B_1 C_3$$

$$A_3 = \varepsilon_{312} B_1 C_2 + \varepsilon_{321} B_2 C_1$$
$$= B_1 C_2 - B_2 C_1$$

A very useful relation is

$$\sum_{i=1}^{3} \varepsilon_{ijk}\, \varepsilon_{ilm} = \delta_{jl}\, \delta_{km} - \delta_{jm}\, \delta_{kl}$$

Since $\varepsilon_{ijk} = 0$ unless ijk are all different the above will be non zero only if the pair j,k has the same numbers as the pair l,m.
when $j=l$ and $k=m$, the above is $(\varepsilon_{ijk})^2 = +1$
when $j=m$ and $k=l$, the above is $\varepsilon_{ijk}\, \varepsilon_{ikj} = -1$
You can check that both sides of the above equation obey these properties, hence the equality

<u>Example:</u> $\vec{A} \times (\vec{B} \times \vec{C})$

ith component of above is

$$\sum_{jklm} \varepsilon_{ijk} A_j \underbrace{\varepsilon_{klm} B_l C_m}_{k^{th} \text{ component of } \vec{B}\times\vec{C}}$$

$$= \sum_{jklm} \varepsilon_{kij}\, \varepsilon_{klm} A_j B_l C_m$$

$$= \sum_{jlm} \left[\delta_{il}\, \delta_{jm} - \delta_{im}\, \delta_{jl} \right] A_j B_l C_m$$

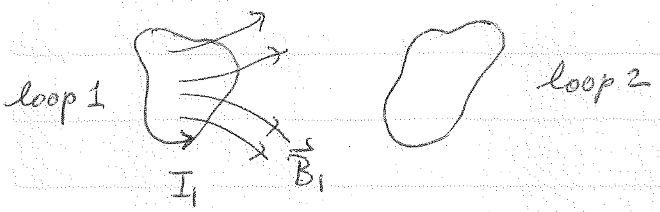
$$= \sum_j A_j B_i C_j - A_j B_j C_i$$

$$= B_i (\vec{A}\cdot\vec{C}) - C_i (\vec{A}\cdot\vec{B})$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

Inductance

Mutual Inductance



$$\vec{A}(\vec{r}_2) = \frac{\mu_0}{4\pi}\int d^3r_1' \frac{\vec{j}_1(\vec{r}_1')}{|\vec{r}_2 - \vec{r}_1|}$$

what is magnetic flux through loop 2, due to current flowing in loop 1 ?

field produced by $\vec{I}_1$

$$\Phi_2 = \int_{S_2} \vec{B}_1 \cdot d\vec{a}_2 = \int_{S_2} (\vec{\nabla}\times\vec{A}_1)\cdot d\vec{a}_2 = \oint_{\Gamma_2} \vec{A}_1 \cdot d\vec{l}_2$$

S_2 surface enclosed by loop 2

in Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$, $\quad \vec{A}_1(\vec{r}_2) = \dfrac{\mu_0}{4\pi}\displaystyle\oint_{\Gamma_1} dl_1 \frac{\vec{I}_1}{|\vec{r}_2 - \vec{r}_1|} = \frac{\mu_0}{4\pi} I \oint \frac{d\vec{l}_1}{|\vec{r}_2 - \vec{r}_1|}$

Γ_1: loop 1

$$\Rightarrow \Phi_2 = \frac{\mu_0}{4\pi} I_1 \oint_{\Gamma_2}\left(\oint_{\Gamma_1} \frac{d\vec{l}_1}{|\vec{r}_2 - \vec{r}_1|}\right)\cdot d\vec{l}_2$$

$$= \frac{\mu_0}{4\pi} I_1 \oint_{\Gamma_1}\oint_{\Gamma_2} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{|\vec{r}_2 - \vec{r}_1|} \equiv M_{21} I_1 \qquad M_{21} = \frac{\mu_0}{4\pi}\oint\oint \frac{d\vec{l}_1\cdot d\vec{l}_2}{|\vec{r}_2 - \vec{r}_1|}$$

mutual inductance of loops 1 and 2.

Similarly, flux through loop 1, due to current I_2 in loop 2 is:

$$\Phi_1 = \frac{\mu_0}{4\pi} I_2 \oint_{\Gamma_2}\oint_{\Gamma_1} \frac{d\vec{l}_2 \cdot d\vec{l}_1}{|\vec{r}_2 - \vec{r}_1|} \equiv M_{12} I_2$$

we see that $M_{12} = M_{21}$

$M_{12} = M_{21} \equiv M$ is a purely geometrical quantity.

Flux through loop 2 when I flows in loop 1
= Flux through loop 1 when I flows in loop 2

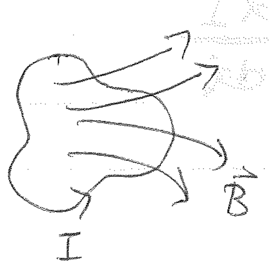
for <u>any</u> two loops.

If vary current in loop 1, flux through loop 2 changes
$\Rightarrow$ emf develops around loop 2

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M\frac{dI_1}{dt}$$

$\Rightarrow$ induced current $I_2 = \dfrac{\mathcal{E}_2}{R_2}$ ← resistance of loop 2
flows in loop 2
when current in loop 1 is changed.
This is the principle behind a transformer.

<u>Self Inductance</u>



What is magnetic flux through loop, due to current flowing in loop?

$$\Phi = \oint_\Gamma \vec{A}\cdot d\vec{l} = \frac{\mu_0 I}{4\pi}\oint_\Gamma\oint_\Gamma \frac{d\vec{l}\cdot d\vec{l}'}{|\vec{r}-\vec{r}'|} \equiv LI$$

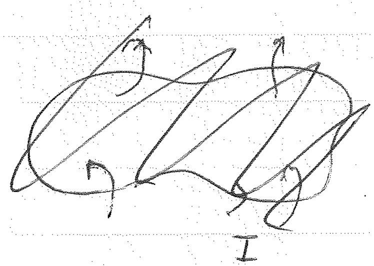
self inductance

both $\vec{r}$ and $\vec{r}'$ lie on same loop Γ.

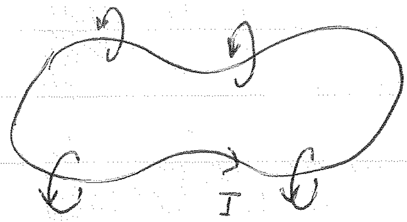
inductance measured in "henries" (H)

$1\ H = 1$ volt-sec/amp

self inductance always positive



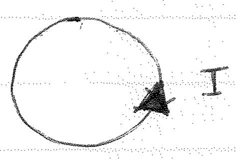
each segment $I \rightarrow$
generates $\vec{B}$ field that circulates around it according to right hand rule



$\Rightarrow$ net flux is always positive for counter clockwise current changing I in loop, changes Φ through loop, creates emf around loop $\quad \mathcal{E} = -\frac{d\Phi}{dt}$

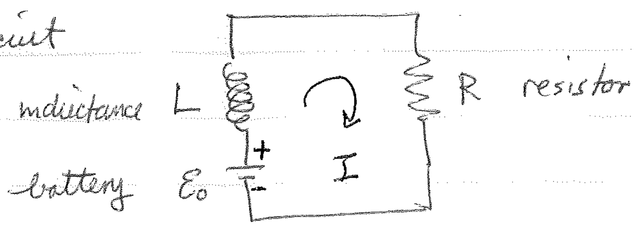
$$\Rightarrow \quad \mathcal{E} = -L\frac{dI}{dt} \qquad\qquad L > 0 \text{ always}$$

this emf $\mathcal{E}$ acts to <u>oppose</u> any change in current – its called the <u>back emf</u>



if I counterclockwise is increased, then $\mathcal{E}$ induced is negative, ie the induced $\mathcal{E}$ tries to drive a current in the opposite (clockwise) direction, to oppose the increase in I

<u>ex:</u> "LR" circuit



total emf in ~~circuit~~ circuit is:

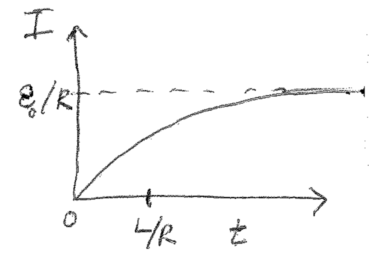
$$\mathcal{E}_0 - L\frac{dI}{dt} = IR \quad \leftarrow \text{ohms law for the resistor}$$

$$\frac{dI}{dt} = -\frac{R}{L}I + \frac{\mathcal{E}_0}{L}$$

1^{st} order differential eqn for $I(t)$.

if switch on battery at $t=0$ $\Big\rangle$ Solution is

$$I(t) = \frac{\mathcal{E}_0}{R}\left(1 - e^{-(R/L)t}\right)$$



current increases to steady state value $\mathcal{E}_0/R$ over time $t \approx L/R$.