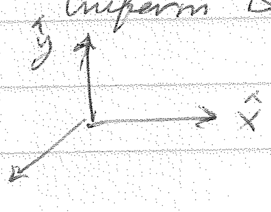
7.7)

Metal bar of mass m slides on frictionless conducting rails as shown in the diagram


uniform $\vec{B}$ into page




$\vec{B} = -B\hat{z}$

a) If bar moves to right with speed v, what is current in resistor? What direction does it flow.

$\vec{v} = v\hat{x}$

We compute the emf induced in the loop

method ① Lorentz force on electrons in moving bar that drives them around the loop is

$$\vec{F}_L = q\,\vec{v}\times\vec{B} = q(v\hat{x})\times(-B\hat{z})$$
$$= qvB\,\hat{y}$$

⇒ current flows counterclockwise

$$\mathcal{E} = \oint \frac{\vec{F}_L}{q}\cdot d\vec{l} = vBl$$

↑ integrate counter clockwise around loop

current in resistor is $\boxed{I = \frac{\mathcal{E}}{R} = \frac{vBl}{R}}$

method ② By Faraday's law  $\Phi = -Blx$

we we compute flux taking normal to loop in $\hat{z}$ direction, and $x$ is distance from sliding bar to the resistor.

Then emf computed counter clockwise around loop is

$$\mathcal{E} = -\frac{d\Phi}{dt} = Bl\frac{dx}{dt} = Blv$$

same as by method ①

b) What is the magnetic force on the bar? In what direction is the force?

Lorentz force on the bar is

$$\vec{F}_{bar} = \int_{bar} I\, d\vec{l} \times \vec{B}$$

$$= I\left(l\hat{y} \times (-B\hat{z})\right)$$

$$\boxed{\vec{F}_{bar} = -IlB\hat{x} = -\frac{vl^2B^2}{R}\hat{x}}$$

$\vec{F}_{bar}$ is directed opposite to direction of motion of the bar

c) If velocity of bar is $v_0$ at $t=0$, what is $v(t)$ at later times?

Newton's Eqn: $m\frac{d\vec{v}}{dt} = \vec{F}_{bar} = -\frac{v\ell^2B^2}{R}\hat{x}$

$$\frac{dv}{dt} = -\frac{\ell^2B^2}{mR} v$$

$$\Rightarrow \boxed{v(t) = v_0 e^{-t/\tau} \quad \text{where } \frac{1}{\tau} = \frac{\ell^2B^2}{mR}}$$

d) The initial kinetic energy was $\frac{1}{2}mv_0^2$. Show that this is the energy dissipated in the resistor from $t=0$ to $t=\infty$ after bar has stopped moving.

Power dissipated in resistor is $\mathcal{E} I$

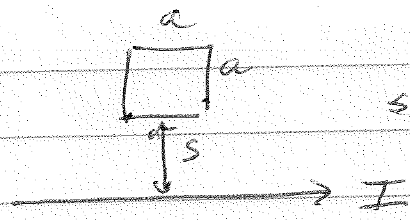
Total energy dissipated is $W = \int_0^\infty dt\, \mathcal{E} I$

$$W = \int_0^\infty dt\, (B\ell v)\left(\frac{vB\ell}{R}\right) = \frac{\ell^2B^2}{R}\int_0^\infty dt\, v^2(t)$$

$$= \frac{\ell^2B^2 v_0^2}{R}\int_0^\infty dt\, e^{-2t/\tau} = \frac{\ell^2B^2v_0^2}{R}\left(-\frac{\tau}{2}\right)\left[e^{-2t/\tau}\right]_0^\infty$$

$$= \frac{\ell^2B^2v_0^2}{R}\,\frac{\tau}{2} = \frac{\ell^2B^2v_0^2}{R}\,\frac{mR}{2\ell^2B^2} = \frac{1}{2}mv_0^2$$

7.8




square loop distance s from current carrying wire

a) What is flux of magnetic field through the loop?

use cylindrical coordinates with $\vec{I} = I\hat{z}$
magnetic field from the wire is $\vec{B}(\vec{r}) = B(r)\hat{\varphi}$

Ampere: $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}$
integrate over circle of radius $r$ $\Rightarrow 2\pi r B(r) = \mu_0 I$

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\varphi}$$

flux through the loop is $\Phi = a\int_a^{a+s} dr \, \frac{\mu_0 I}{2\pi r} = \frac{\mu_0 I a}{2\pi} \ln\left(\frac{a+s}{a}\right)$
(computing flux out of page)

b) If loop is pulled away from wire with speed $v$, what is the emf around the loop? In what direction will the induced current flow?

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

Since loop is pulled away from the wire, the flux decreases, so $\mathcal{E} > 0$, so current flows

<u>counterclockwise</u>

$\mathcal{E} = \frac{\mu_0 I}{2\pi} \frac{d}{dt}\left(s \ln\left(1+\frac{s}{a}\right)\right) = \frac{\mu_0 I}{2\pi}\left[\ln\left(1+\frac{s}{a}\right) + \frac{s\left(\frac{1}{a}\right)}{1+\frac{s}{a}}\right.$

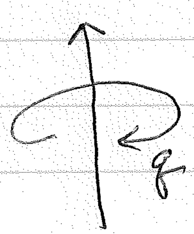
$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{\mu_0 I a}{2\pi} \frac{d}{dt}\left[\ln\left(1+\frac{a}{s}\right)\right]$$

$$= -\frac{\mu_0 I a}{2\pi} \frac{\left(-\frac{a}{s^2}\right)\frac{ds}{dt}}{1+a/s} = \frac{\mu_0 I a^2 v}{2\pi\,(s^2+sa)}$$

c) what if loop is pulled to the right at speed $v$

now $\frac{d\Phi}{dt} = 0$, no current flows

**Betatron**  use $\frac{\partial \vec{B}}{\partial t}$ to accelerate a charge in a circular orbit



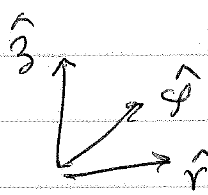

suppose have a magnetic field $\vec{B}(\vec{r}) = B(r)\hat{z}$ rotationally symmetric about z axis

A charged particle q with velocity $\vec{v}$ will travel in a circular orbit of radius r given by

$$m\vec{a} = m\frac{d\vec{v}}{dt} = q\,\vec{v}\times B(r)\hat{z}$$

$$= -m\frac{v^2}{r}\hat{r} = q\,v\,B(r)\left(\hat{\varphi}\times\hat{z}\right)$$

↑ centripetal accell  $\vec{v} = v\hat{\varphi}$ in cylindrical coords




$$\hat{\varphi}\times\hat{z} = \hat{r}$$

$$-\frac{mv^2}{r}\hat{r} = q\,v\,B(r)\,\hat{r}$$

$$mv = -q\,r\,B(r)$$

the (−) sign says that $\vec{v}$ travels in the $-\hat{\varphi}$ direction ie the charge goes <u>clockwise</u> around the $\hat{z}$ axis.

Hence forth lets use $v = |\vec{v}|$ and $\vec{v} = -v\hat{\varphi}$

$$\boxed{mv = qrB(r)}$$ determines the relation between $v$ and $r$ for the cyclotron motion of $q$.

Now suppose $B(r)$ changes with time then $\Rightarrow$ magnetic flux through the orbit of the charge $q$ will change $\Rightarrow$ there will be an induced $\vec{E}$ that will accelerate the charge.

We want to find a condition on $B(r)$ so that the induced $\vec{E}$ will accelerate $q$, BUT keep it in orbit at the same radius $r$.

First we compute the induced $\vec{E}$.

For $\vec{B} = B(r)\hat{z}$, by symmetry we have
$\vec{E}(\vec{r}) = E(r)\hat{\varphi}$

[ we know this because $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ has the same form as $\vec{\nabla}\times\vec{B} = \mu_0\vec{j}$, and we know that a $\vec{j}(\vec{r}) = j(r)\hat{z}$, like a current flowing down a straight wire, will result in a $B(r)\hat{\varphi}$ curling around the wire ]

So now solve for $E(r)$

For $\vec{E}(\vec{r}) = E(r)\,\hat{\varphi}$ we have

$$\vec{\nabla}\times\vec{E} = \frac{1}{r}\frac{d}{dr}\left(r E(r)\right)\hat{z} \quad \text{in cylindrical coords}$$

$$= -\frac{\partial B(r)}{\partial t}\,\hat{z}$$

$$\Rightarrow \frac{d}{dr}\left(rE(r)\right) = -r\frac{\partial B(r)}{\partial t}$$

integrate both sides from $r'=0$ to $r'=r$
and multiply by $2\pi$

$$2\pi\int_0^r dr'\,\frac{d}{dr'}\left(r'E(r')\right) = -2\pi\int_0^r dr'\,r'\,\frac{\partial B(r')}{\partial t}$$

The left hand side is just $2\pi r E(r)$
the right hand side is

$$2\pi\int_0^r dr'\,r'\,\frac{\partial B(r')}{\partial t} = \frac{d}{dt}\Bigg[\underbrace{\int_0^{2\pi} d\varphi \int_0^r dr'\,r'\,B(r')}_{\text{flux of B through orbit of radius } r}\Bigg]$$

$$= \frac{d\Phi}{dt}$$

So $\quad 2\pi r E(r) = -\dfrac{d\Phi}{dt}$

Note: regarding the orbit of $q$ as a circular path, $2\pi r\,E(r) = \mathcal{E}$ is just the emf around that path! so equ for E is really the same as $\mathcal{E} = -\frac{d\Phi}{dt}$

Let $\Phi = \pi r^2 B_{av}$ where $B_{av}$ is the average magnetic field over the circle bounded by the orbit of $q$

then:

$$2\pi r E(r) = -\pi r^2 \frac{dB_{av}}{dt}$$

$$\boxed{E(r) = -\frac{r}{2}\frac{dB_{av}}{dt}}$$

Note, if $\frac{\partial B_{av}}{\partial t} > 0$ then $E < 0$ ie $E$ is in the $-\hat{\varphi}$ direction

Since the charge $q$ has $\vec{v} = -v\hat{\varphi}$ moving in the clockwise $-\hat{\varphi}$ direction, the induced $\vec{E}$ is in the same direction as $\vec{v}$ and so will accelerate the charge.

Now, if we assume the charge keeps moving in the same orbit of radius $r$, then Newton equation for the $(-\hat{\varphi})$ component of the motion is

$$m\frac{dv}{dt} = qE = q\frac{r}{2}\frac{dB_{av}}{dt}$$

↑ $\vec{v}$ is in $-\hat{\varphi}$ direction

↑ $E$ is in $-\hat{\varphi}$ direction

So
$$\boxed{m\frac{dv}{dt} = q\frac{r}{2}\frac{dB_{av}}{dt}}$$

But we assumed that the charge stayed in the same orbit of radius r, so Newton's equation for the $\hat{r}$ component of motion must give the same cyclotron condition as before, ie,

$$mv = qr\,B(r)$$

$$\Rightarrow \boxed{m\frac{dv}{dt} = qr\frac{\partial B(r)}{\partial t}}$$ (since r is constant by assumption)

These two equations can only both be true if

$$\frac{\partial B(r)}{\partial t} = \frac{1}{2}\frac{\partial B_{av}(r)}{\partial t}$$

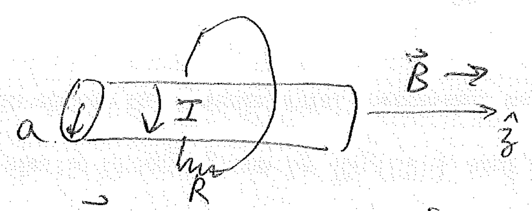
or, if $\vec{B}$ increases with time by the same factor everywhere in space

$$B(r) = \tfrac{1}{2}B_{av}(r)$$

The magnetic field at radius r of the charges orbit must be $\frac{1}{2}$ the average of the magnetic field averaged over the circle bounded by the orbit,

~~7.14~~

7.17




Long solenoid of radius a
N turns per length, & looped by wire of radius R

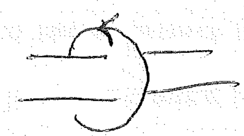
$\vec{B} = \mu_0 N I \hat{z}$ inside solenoid

$\frac{dB}{dt} = \mu_0 N \frac{dI}{dt} \hat{z}$ $\qquad \frac{dI}{dt} = k$

flux only goes through area of solenoid $\pi a^2$

a) $\frac{d\Phi}{dt} = \pi a^2 \mu_0 N k = -\mathcal{E}$ $\qquad$ current flows

$\frac{\mathcal{E}}{R} = I = \frac{\pi a^2 \mu_0 N k}{R}$



clockwise since $\mathcal{E} < 0$

b) Suppose solenoid taken out & reversed
What total charge Q passes through the resistor R?

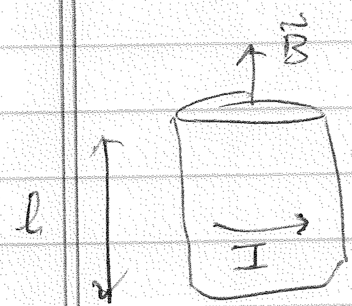
$\mathcal{E} = -\frac{d\Phi}{dt}$

$I = \frac{\mathcal{E}}{R} = \frac{-d\Phi}{R\,dt}$

total charge is $Q = \int_0^T dt\, I = \frac{-1}{R} \int_0^T \frac{d\Phi}{dt} = \frac{-1}{R}\left[\Phi_f - \Phi_i\right]$

$Q = \frac{-\pi a^2}{R}\left[-\mu_0 N I - \mu_0 N I\right] = \frac{2\mu_0 N I \pi a^2}{R}$

Self inductance of a solenoid length $l$, radius $R$




$$\vec{B} = \mu_0 N I \hat{z}$$

$N$ ← # turns of wire per unit length

Total flux through all the wire loops that make up the solenoid is

$$\Phi = (\mu_0 N I \pi R^2)(N l) = \mu_0 N^2 l \pi R^2 I$$

(flux through one loop) (number of loops)

$\Phi = L I$ defines self inductance $L$

$$\rightarrow \boxed{L = \mu_0 N^2 l \pi R^2}$$

Another way to do the calculation:

The energy stored in this magnetostatic configuration is

$$W_{mag} = \frac{1}{2\mu_0} \int d^3r \, |\vec{B}|^2 = \frac{1}{2\mu_0} (\mu_0 N I)^2 (\pi R^2 l)$$

↑ B-field in solenoid ↑ volume inside solenoid

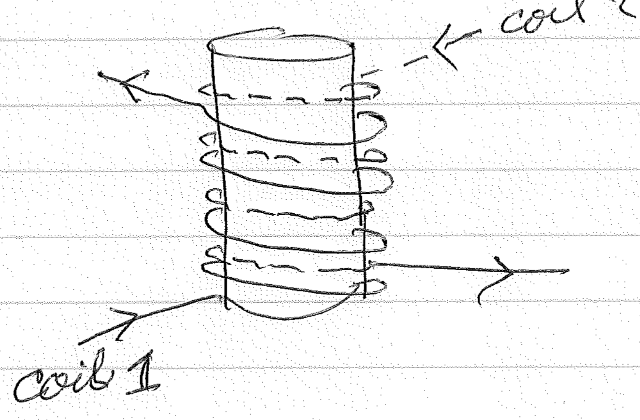
(we assume only B we need consider is that inside the solenoid)

$$W_{mag} = \frac{1}{2} \mu_0 N^2 l \pi R^2 I^2$$

But we also know $W_{mag} = \frac{1}{2} L I^2 \Rightarrow \boxed{L = \mu_0 N^2 l \pi R^2}$

same result as above

# Transformer




coil 1 with $N_1$ turns per length
coil 2 with $N_2$ turns per length
wrapped around same solenoid

$\Phi$ is the flux through each turn of either coil

If $\Phi$ changes in time, it induces emf in coil 1

$$\mathcal{E}_1 = -N_1 \frac{d\Phi}{dt}$$

and induces emf in coil 2

$$\mathcal{E}_2 = -N_2 \frac{d\Phi}{dt}$$

Since $\Phi$ is the same for both coils we then get

$$\frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{N_2}{N_1}$$

If we fix voltage in coil 1 to be $\mathcal{E}_1$, then the above transformer will induce voltage $\mathcal{E}_2 = \frac{N_2}{N_1}\mathcal{E}_1$ in coil 2.

If $N_2 > N_1$ then voltage is "stepped up"