Maxwell's Equations

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho \qquad \vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

Ampere's law for magnetostatics was

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

but $\underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B})}_{=0} = \mu_0\vec{\nabla}\cdot\vec{J} = \underbrace{-\mu_0\frac{\partial\rho}{\partial t}}_{\neq 0}$ by charge conservation

$= 0$ by general theorem of vector calculus

$\neq 0$ unless have electrostatics + magnetostatics

⇒ Ampere's law can't be valid outside static situations

To fix: write $-\mu_0\frac{\partial\rho}{\partial t} = -\mu_0\varepsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E}) = \vec{\nabla}\cdot\left(-\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$

$\varepsilon_0\vec{\nabla}\cdot\vec{E}=\rho$

correction is:

So $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\cdot\left(\mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right) = 0$

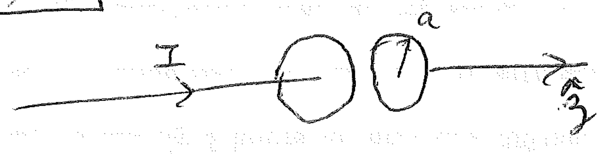
$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \underbrace{\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}}_{\text{"displacement current"}}$$

← Maxwell's correction to Ampere's Law

$$\oint_P \vec{B}\cdot d\vec{l} = \mu_0 I_{encl} + \mu_0\varepsilon_0\int_S\left(\frac{\partial\vec{E}}{\partial t}\right)\cdot d\vec{a}$$

so $\frac{\partial\vec{E}}{\partial t}$ is a source of $\vec{B}$, just like $\frac{\partial\vec{B}}{\partial t}$ is a source of $\vec{E}$

~~7.31~~ 7.35



constant I.

$$\nabla \times B = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}$$

$$\oint d\vec{l} \cdot \vec{B} = \mu_0 I_{encl} + \mu_0 \epsilon_0 \int d\vec{a} \cdot \frac{\partial \vec{E}}{\partial t}$$

find B between plates for $r < a$.

charge on plates $Q = It$ on left plate, $-It$ on right plate

$\Rightarrow$ E between plates is $\vec{E} = \frac{\sigma}{\epsilon_0} \hat{z} = \frac{Q}{\pi a^2 \epsilon_0} \hat{z}$

parallel plate capacitor

$\vec{E} = \frac{It}{\epsilon_0 \pi a^2} \hat{z}$, $\frac{\partial \vec{E}}{\partial t} = \frac{I}{\epsilon_0 \pi a^2} \hat{z}$ like a current flowing in $\hat{x}$ direction.

in region between plates, symmetry $\Rightarrow$ $\vec{B} = B(r) \hat{\phi}$

Take loop of radius r centered about wire, inbetween plates

$$\oint d\vec{l} \cdot \vec{B} = 2\pi r B(r) = \mu_0 \underset{\overset{\parallel}{0}}{I_{encl}} + \mu_0 \epsilon_0 \int d\vec{a} \cdot \frac{I}{\epsilon_0 \pi a^2} \hat{z}$$

area of loop $= \pi r^2$

$$= \mu_0 \epsilon_0 \frac{\pi r^2 I}{\epsilon_0 \pi a^2}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0 r}{2\pi a^2} I \hat{\phi}$$

just like we had a wire with uniform current density $\vec{J} = \frac{I}{\pi a^2} \hat{x}$

for $r > a$, if ignore "edge" effects from non uniformity of $\vec{E}$ at edges of plates

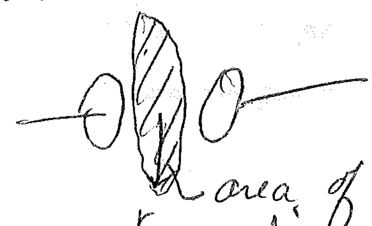
$$\oint d\vec{\ell} \cdot \vec{B} = 2\pi r\, B(r) = \underset{\overset{\shortparallel}{0}}{\mu_0 I_{encl}} + \cancel{\mu_0\varepsilon_0 \int d\vec{a} \cdot \frac{I}{\varepsilon_0 \pi a^2}\hat{x}}$$

$$+ \mu_0\varepsilon_0 (\pi a^2) \frac{I}{\varepsilon_0 \pi a^2}$$

since only area between plates has $\frac{\partial E}{\partial t} \neq 0$

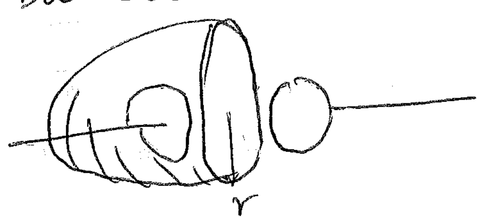
$$= \mu_0 I$$

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{2\pi r}\hat{\phi}$$ just like around wire with current I

to do above, took area of loop as 

area of integration

but could also take any area bounded by curve,



now $\int d\vec{a} \cdot \frac{\partial \vec{E}}{\partial t} = 0$ on this area

but, $I_{encl} = I$

So $B = \frac{\mu_0 I \hat{\phi}}{2\pi r}$ as before

we need Maxwell's displacement current to get consistent results! ie independent of what surface is used for the integration

Energy and Momentum Conservation

in electrostatics $W_{elec} = \frac{\varepsilon_0}{2} \int d^3r \, E^2$

in magnetostatics $W_{mag} = \frac{1}{2\mu_0} \int d^3r \, B^2$

Now we treat the full electrodynamic situation

Maxwell's Equations

$$\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0 \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Consider the power dissipated by an electric current flowing in a wire

power $= VI$

V is voltage drop $= \mathcal{E}$ emf

I is current, charge/time

$V = EL$

E in wire

length of wire

$I = Aj$

current density

cross sectional area

$$\text{power} = VI = (EL)(Aj) = (Ej)(LA)$$

$$= (Ej)(\text{volume})$$

If energy is conserved, then this energy dissipated by the flowing current must go somewhere.

Where does it go? Goes into heating up the wire "Joule heating". What does that heat correspond to mechanically? Increased kinetic energy of the particles in the wire!

Let W_{mech} be the total mechanical energy of particles in some volume of space

$$W_{mech} = (\text{kinetic energy} + \text{potential energy})$$

then expect $$\boxed{\frac{dW_{mech}}{dt} = \int_{vol} d^3r\, \vec{E}\cdot\vec{J}}$$

Another way: Let W_{mech} is the mechanical energy of a collection of charged particles. Then the work-energy theorem of mechanics says the change in energy of a charge q is given by the forces acting on q,

$$dW = \vec{F}\cdot d\vec{r} = (q\vec{E} + q\vec{v}\times\vec{B})\cdot d\vec{r}$$

$$\frac{dW}{dt} = \vec{F}\cdot\frac{d\vec{r}}{dt} = (q\vec{E} + q\vec{v}\times\vec{B})\cdot\vec{v}$$

$$= q\vec{v}\cdot\vec{E}$$

Now add up over all charges in the volume

$$\frac{dW_{mech}}{dt} = \sum_{q_i \text{ in vol}} q_i \vec{v}_i \cdot E(\vec{r}_i)$$

$$= \int_{vol} d^3r \ \vec{J}(\vec{r}) \cdot \vec{E}(\vec{r})$$

where we used $\vec{J}(\vec{r}) = \sum_i q_i \vec{v}_i \, \delta(\vec{r} - \vec{r}_i)$

$$\boxed{\frac{dW_{mech}}{dt} = \int_{vol} d^3r \ \vec{J} \cdot \vec{E}}$$

$$\frac{dW_m}{dt} = \int d^3r\, \vec{J}\cdot\vec{E}$$

Ampere's Law $\vec{J} = \frac{\vec{\nabla}\times\vec{B}}{\mu_0} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

$$\frac{dW_m}{dt} = \int d^3r \left[\underbrace{\frac{\vec{E}\cdot(\vec{\nabla}\times\vec{B})}{\mu_0}}_{\substack{\text{integrate}\\ \text{by parts}}} - \underbrace{\varepsilon_0 E \cdot \frac{\partial \vec{E}}{\partial t}}_{= \frac{1}{2}\frac{\partial E^2}{\partial t}} \right]$$

$$\vec{\nabla}\cdot(\vec{E}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{B})$$

$$\frac{dW_m}{dt} = \int d^3r \left[\frac{1}{\mu_0}\vec{B}\cdot(\vec{\nabla}\times\vec{E}) - \frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B}) - \varepsilon_0 \frac{1}{2}\frac{\partial E^2}{\partial t} \right]$$

use $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ Faraday

use $\vec{B}\cdot\frac{\partial \vec{B}}{\partial t} = \frac{1}{2}\frac{\partial B^2}{\partial t}$

$$= \int d^3r \left[\left(\frac{-1}{2}\right)\left(\frac{\partial B^2}{\mu_0 \partial t} + \varepsilon_0 \frac{\partial E^2}{\partial t}\right) - \frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B}) \right]$$

$$\frac{dW_m}{dt} = -\frac{d}{dt}\int_{Vol} d^3r \left(\frac{1}{2\mu_0}B^2 + \frac{\varepsilon_0}{2}E^2\right) - \frac{1}{\mu_0}\oint_{Surface} d\vec{a}\cdot(\vec{E}\times\vec{B})$$

define $W_{EB} = \int_{Vol} d^3r \left(\frac{1}{2\mu_0}B^2 + \frac{\varepsilon_0}{2}E^2\right)$ electro-magneto energy in volume V

$\vec{S} \equiv \frac{\vec{E}\times\vec{B}}{\mu_0}$ "Poynting vector"

= energy density current

$$\frac{dW_m}{dt} = -\frac{dW_{EB}}{dt} - \oint d\vec{a} \cdot \vec{S}$$

increase in mechanical energy (kinetic energy of charges) = energy lost from $\vec{E} + \vec{B}$ fields − energy from $E + B$ fields flowing out of volume through surface

write $u_{EB} = \frac{\varepsilon_0}{2} E^2 + \frac{B^2}{2\mu_0}$ energy density of electromagnetic fields

u_M = mechanical energy density

$$\frac{d}{dt}\int_V d^3r\, u_m + \frac{d}{dt}\int_V d^3r\, u_{EB} = -\oint_S \vec{S} \cdot d\vec{a} = -\int_V d^3r\, \vec{\nabla} \cdot \vec{S}$$

$$\frac{\partial}{\partial t}\left(u_M + u_{EB}\right) = -\vec{\nabla} \cdot \vec{S}$$

law of local conservation of energy for e-m fields

(Same form as charge conservation $\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{j}$

$\vec{S}$ is flux of energy carried by $\vec{E} + \vec{B}$ fields

$\oint_S \vec{S} \cdot d\vec{a}$ is energy per unit time carried by $\vec{E} + \vec{B}$ fields through surface S