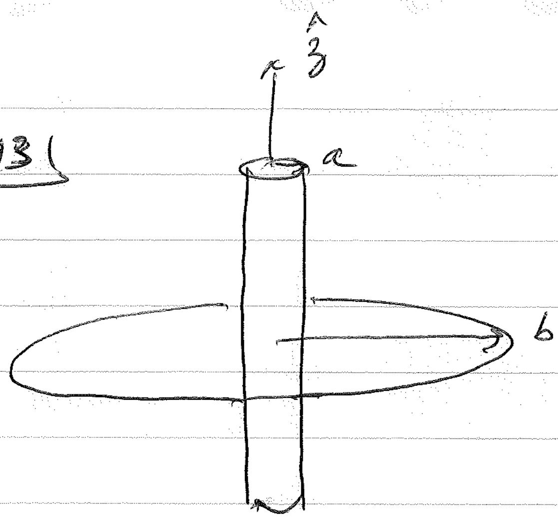
8.13)



solenoid of radius a, surrounded by a circular wire ring of radius b

$a \ll b$

solenoid has current I_s flowing in N turns of wire per unit length

a) $\vec{B}$ field from solenoid is

$$\vec{B} = \begin{cases} \mu_0 N I_s \hat{z} & \text{for } r<a \\ 0 & \text{for } r>a \end{cases}$$

Flux through circular wire is

$$\Phi = \pi a^2 B = \mu_0 \pi a^2 N I_s$$

If Φ changes (because I_s changes) then there is an emf $\mathcal{E}$ induced in the circular wire ring

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\mu_0 \pi a^2 N \frac{dI_s}{dt}$$

If the circular wire ring has a total resistance R then the current induced in the ring is

$$\vec{I}_r = \frac{\mathcal{E}}{R}\hat{\varphi} = -\frac{\mu_0 \pi a^2 N}{R}\frac{dI_s}{dt}\hat{\varphi}$$

Note: Φ computed in $+\hat{z}$ direction $\Rightarrow$ $\mathcal{E}$ computed in the $+\hat{\varphi}$ (counterclockwise) direction (right hand rule)

$\Rightarrow \vec{I}_r$ is in the $\hat{\varphi}$ direction

If $\frac{dI_s}{dt} > 0$ then $\mathcal{E} < 0 \Rightarrow I_r$ is flowing in the $-\hat{\varphi}$ direction, i.e. flowing clockwise.

b) power dissipated in ring is $P = I_r^2 R = I_r \mathcal{E}$

where does this power come from? It must be coming from the solenoid!

Compute the flux of energy flowing away from the solenoid.

energy flux given by the Poynting vector

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

we want to evaluate this on the outside surface of the solenoid.

The $\vec{E}$ field in this $\vec{S}$ is just the $\vec{E}$ field induced by Faradays Law. By symmetry we expect

$\vec{E}(\vec{r}) = E(r)\hat{\varphi}$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int d\vec{a} \cdot \vec{B}$$

for a circular path of radius r we then get

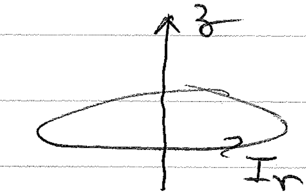
$$2\pi r E(r) = -\frac{d\Phi}{dt} = \mathcal{E} \leftarrow \text{the emf in the ring}$$

So we can write

$$\vec{E}(\vec{r}) = \frac{\mathcal{E}}{2\pi r}\,\hat{\varphi} \qquad \mathcal{E} = -\mu_0 \pi a^2 N \frac{dI_s}{dt} \text{ from (a)}$$

What is the $\vec{B}$ that appears in $\vec{S}$? It is NOT the $\vec{B}$ field from the solenoid as that is zero outside the solenoid. Rather the $\vec{B}$ outside the solenoid must be the $\vec{B}$ produced by the current I_r in the circular ring!

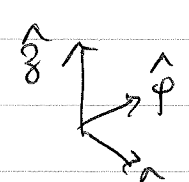
From example 5-6 in the text we have for $\vec{B}$ along the $\hat{z}$ from a circular ring



$$\vec{B}(z) = \frac{\mu_0 I_r}{2} \frac{b^2\,\hat{z}}{(b^2+z^2)^{3/2}}$$

We really want $\vec{B}$ at $r=a$ on the surface of the solenoid, but since $a \ll b$ it will be a good enough approximation to use the above $\vec{B}$ at $r=0$ along the z axis.

So now $$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\underbrace{\left(\frac{\mathcal{E}}{2\pi a}\right)}_{E(r=a)}\underbrace{\left(\frac{\mu_0 I_r b^2}{2(b^2+z^2)^{3/2}}\right)}_{B}\hat{\varphi}\times\hat{z}$$



$\hat{\varphi}\times\hat{z} = \hat{r}$

$$\vec{S} = \frac{\mathcal{E} I_r b^2}{2\pi a}\,\frac{1}{2(b^2+z^2)^{3/2}}\,\hat{r}$$

on surface of solenoid.

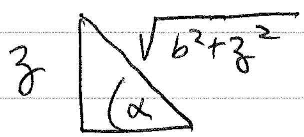
To get the total energy per unit time flowing away from the solenoid we integrate the flux of $\vec{S}$ through the surface of the solenoid.

$$P = \int d\vec{a} \cdot \vec{S} = \int_{-\infty}^{\infty} dz \int_0^{2\pi} d\varphi\, a\, \hat{r} \cdot \vec{S}$$

$$= \mathcal{E} I_r \frac{b^2}{2\pi a}\, 2\pi a \int_{-\infty}^{\infty} dz \frac{1}{2(b^2+z^2)^{3/2}}$$

$$= \mathcal{E} I_r \int_{-\infty}^{\infty} dz \frac{b^2}{2(b^2+z^2)^{3/2}}$$

to do the integral one makes a trig substitution



$$b \tan\alpha = z \Rightarrow dz = \frac{b}{\cos^2\alpha}\, d\alpha$$

$$\frac{1}{\sqrt{z^2+b^2}} = \frac{\cos\alpha}{b}$$

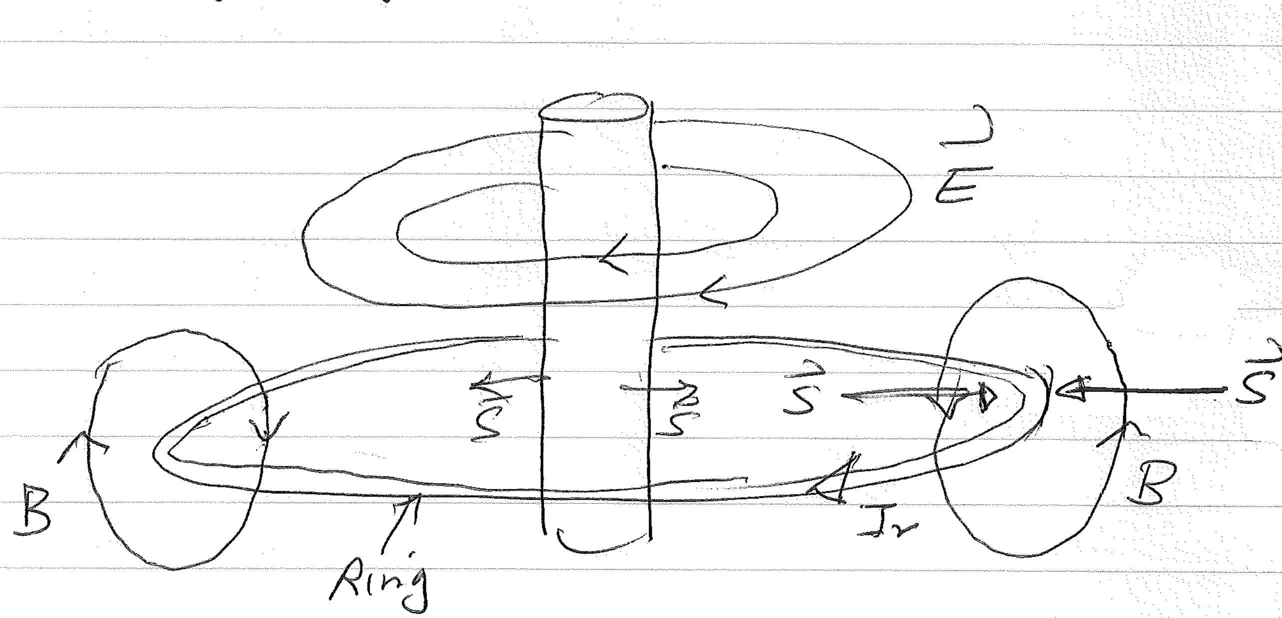
$$\int_{-\infty}^{\infty} dz \frac{1}{(b^2+z^2)^{3/2}} = \int_{-\pi/2}^{\pi/2} d\alpha \frac{b}{\cos^2\alpha} \frac{\cos^3\alpha}{b^3} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\alpha \frac{\cos\alpha}{b^2}$$

$$= \frac{2}{b^2}$$

So $\displaystyle\int_{-\infty}^{\infty} dz \frac{b^2}{2(b^2+z^2)^{3/2}} = 1$ and $P = \mathcal{E} I_r$

power leaving solenoid = power dissipated in ring!

More generally



At surface of ring, one can see that $\vec{S}$ is always directed inward into the ring.

Momentum Conservation

want similar conservation law for mechanical & electromagnetic momentum

$$\frac{\partial}{\partial t}\left(p_{mi} + p_{EBi} \right) = \vec{\nabla}\cdot\vec{T}_i \qquad i = x, y, z$$

p_{mi} = i^{th} component of mechanical momentum density

p_{EBi} = i^{th} component of electromagnetic momentum density

$\vec{T}_i$ = $-$flux density (or "current") of i^{th} component of momentum density

Since $\vec{T}_i$ is a vector with 3 components, and there are three such vectors, for $i = x, y, z$, we will see that these 3 vectors form the components of a 3×3 tensor (ie matrix)

mechanical momentum density given by Newtons Law $\Big\}$ $\dfrac{\partial \vec{p}_m}{\partial t} = \vec{f} = \rho\vec{E} + \vec{j}\times\vec{B}$

force density $\vec{f} = \rho\vec{E} + \vec{j}\times\vec{B} = \varepsilon_0(\vec{\nabla}\cdot\vec{E})\vec{E} + \left(\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B}$

Now apply vector algebra + Maxwell's equ (see text sec 8.2.2) to manipulate into the form

$$\vec{f} = \varepsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E}\right] + \frac{1}{\mu_0}\left[(\vec{\nabla}\cdot\vec{B})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{B}\right]$$
$$- \frac{1}{2}\vec{\nabla}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \varepsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) \qquad \cancel{(7.91)}\ (8.15)$$

$\vec{f}$ above looks like mess, but it simplifies if one introduces the following 3x3 matrix, known as the "Maxwell Stress Tensor"

$$T_{ij} = \varepsilon_0 (E_i E_j - \tfrac{1}{2}\delta_{ij} E^2) + \frac{1}{\mu_0}(B_i B_j - \tfrac{1}{2}\delta_{ij} B^2)$$

$$i = x, y, z \quad \text{and} \quad \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$\overleftrightarrow{T} = \begin{bmatrix} \varepsilon_0 (E_x^2 - \frac{1}{2}E^2) + \frac{1}{\mu_0}(B_x^2 - \frac{1}{2}B^2) & \varepsilon_0 E_x E_y + \frac{1}{\mu_0} B_x B_y & \varepsilon_0 E_x E_z + \frac{1}{\mu_0} B_x B_z \\ \varepsilon_0 E_x E_y + \frac{1}{\mu_0} B_x B_y & \varepsilon_0 (E_y^2 - \frac{1}{2}E^2) + \frac{1}{\mu_0}(B_y^2 - \frac{1}{2}B^2) & \varepsilon_0 E_y E_z + \frac{1}{\mu_0} B_y B_z \\ \varepsilon_0 E_x E_z + \frac{1}{\mu_0} B_x B_z & E_y E_z + \frac{1}{\mu_0} B_y B_z & \varepsilon_0 (E_z^2 - \frac{1}{2}E^2) + \frac{1}{\mu_0}(B_z^2 - \frac{1}{2}B^2) \end{bmatrix}$$

$T_{ij} = T_{ji} \Rightarrow T$ is symmetric

$$(\vec{\nabla} \cdot \overleftrightarrow{T}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

$$(\vec{\nabla} \cdot \overleftrightarrow{T})_j = \sum_i \frac{\partial}{\partial x_i} T_{ij} = \sum_i \Big[\varepsilon_0 \Big(\frac{\partial E_i}{\partial x_i} E_j + E_i \frac{\partial E_j}{\partial x_i} - \frac{1}{2} \frac{\partial E^2}{\partial x_i} \delta_{ij} \Big) + \frac{1}{\mu_0} \Big(\frac{\partial B_i}{\partial x_i} B_j + B_i \frac{\partial B_j}{\partial x_i} - \frac{1}{2} \frac{\partial B^2}{\partial x_i} \delta_{ij} \Big) \Big]$$

$$(\vec{\nabla} \cdot \overleftrightarrow{T}) = \varepsilon_0 \Big[(\vec{\nabla} \cdot \vec{E})\vec{E} + (\vec{E} \cdot \vec{\nabla})\vec{E} - \tfrac{1}{2} \vec{\nabla} E^2 \Big]$$

$$+ \frac{1}{\mu_0} \Big[(\vec{\nabla} \cdot \vec{B})\vec{B} + (\vec{B} \cdot \vec{\nabla})\vec{B} - \tfrac{1}{2} \vec{\nabla} B^2 \Big]$$

$$= \vec{f} + \varepsilon_0 \frac{\partial}{\partial t}(\vec{E} \times \vec{B}) = \vec{f} + \varepsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t}$$

$$\vec{f} = \frac{\partial \vec{p}_M}{\partial t} = -\varepsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t} + \vec{\nabla} \cdot \overleftrightarrow{T}$$

$$\frac{\partial}{\partial t}\left[\vec{p}_M + \varepsilon_0 \mu_0 \vec{S}\right] = \vec{\nabla} \cdot \overleftrightarrow{T}$$ this is the desired conservation law for momentum

$$\Rightarrow \boxed{\varepsilon_0 \mu_0 \vec{S} = \vec{p}_{EM}}$$ electromagnetic momentum density

~~$-\vec{\nabla}\cdot\overleftrightarrow{T}$ = current or flux~~

$-\overleftrightarrow{T}_{ij}$ is i^{th} component of current, of j^{th} component of electromagnetic momentum density

ie the vector $-\begin{pmatrix} T_{xx} \\ T_{yx} \\ T_{zx} \end{pmatrix}$ is the current for x-component $\vec{p}_{EM\,x}$ of E-M momentum

integral form: $$\int_{Vol} d^3r \left[\frac{\partial}{\partial t}\vec{p}_M + \frac{\partial}{\partial t}\vec{p}_{EB}\right] = \frac{d}{dt}\int_{Vol} d^3r \underbrace{\left(\vec{p}_M + \vec{p}_{EB}\right)}_{\text{total mechanical + electromagnetic field momentum contained in Vol}}$$

$$= \int_{Vol} d^3r\, \vec{\nabla}\cdot\overleftrightarrow{T} = \underbrace{\oint_S d\vec{a}\cdot\overleftrightarrow{T}}_{(-)\text{ flux of field momentum out through surface } S \text{ bounding Vol}}$$

Or we can write

$$\frac{d}{dt}\int_{vol} d^3r\, \vec{p}_{mech} = \frac{d\vec{P}_{mech}}{dt} = -\frac{d}{dt}\int_{vol} d^3r\, \vec{p}_{ER} + \oint_S d\vec{a}\cdot\overleftrightarrow{T}$$

$$\frac{d\vec{P}_{mech}}{dt} = \underbrace{-\frac{d\vec{P}_{EB}}{dt} + \oint_S d\vec{a}\cdot\overleftrightarrow{T}}_{\text{total electromagnetic force on the volume}}$$

$$\vec{F}_{EB} = \frac{d\vec{P}_{mech}}{dt}$$

For a situation where $\vec{E}$ and $\vec{B}$ are constant in time,

$$\frac{d\vec{P}_{EB}}{dt} = 0 \quad \text{and so}$$

$$\frac{d\vec{P}_{mech}}{dt} = \oint_S d\vec{a}\cdot\overleftrightarrow{T}$$ ← gives total electromagnetic force on the volume

This is why $\overleftrightarrow{T}$ is called the Maxwell stress tensor

It is like a pressure acting on the walls of the volume.

Numerically compute $\varepsilon_0\mu_0$, find $\varepsilon_0\mu_0 = \frac{1}{c^2}$ with c = speed of light

$\vec{S} = c^2 \vec{p}_{EB}$

$\uparrow$ momentum density

$\uparrow$ energy current

Suppose energy current is made of "particles that travel with velocity $\vec{c}$. Then $\vec{S} = \vec{c}\, u_{EB}$ u_{EB} is energy density

$u_{EB} = c\, p_{EB}$ = energy-moment relation for photons.

Also can do same for angular momentum

$\vec{\mathcal{L}}_{EB} = \vec{r} \times \vec{p}_{EB} = \varepsilon_0 \vec{r} \times (\vec{E} \times \vec{B})$

= angular momentum density contained in $\vec{E}$ and $\vec{B}$ fields

see Griffiths Sec 8.2-4

Magnetic monopoles Sec 7.3.4

Maxwell's equations: $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ $\vec{\nabla}\cdot\vec{B} = 0$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Maxwell's equations would have a much more symmetric look to them, if one imagined that there were such things as magnetic monopoles (ie. magnetic charges)

$\vec{\nabla}\cdot\vec{B} = 0$ is purely expt result. Suppose we found a magnetic monopole, so that we would now have

$$\vec{\nabla}\cdot\vec{B} = \mu_0 \eta$$

η = volume density of magnetic charge.

$\eta = \sum_i g_i \, \delta(\vec{r}-\vec{r}_i)$ for point monopoles

A point magnetic monopole g would produce a magnetic field $\vec{B} = \frac{\mu_0}{4\pi}\frac{g}{r^2}\hat{r}$

There would be conservation law of magnetic charge $\vec{\nabla}\cdot\vec{k} = -\frac{\partial \eta}{\partial t}$ where $\vec{k}$ is the magnetic charge current density.

Then Faraday's Law would have to be fixed, like Maxwell fixed Ampere's Law

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t}\right) = 0 + \frac{\partial\, \vec{\nabla}\cdot\vec{B}}{\partial t} = \mu_0 \frac{\partial \eta}{\partial t} = -\mu_0 \vec{\nabla}\cdot\vec{k}$$

New Faradays law would be $\vec{\nabla}\times\vec{E} = -\mu_0 \vec{k} - \frac{\partial \vec{B}}{\partial t}$

Now Maxwells equations would look symmetric!
$$\left.\right\} \quad \vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \qquad \vec{\nabla}\cdot\vec{B} = \mu_0 \eta$$
$$\vec{\nabla}\times\vec{E} = -\mu_0 \vec{k} - \frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$