Maxwells Equations in Potential Form

$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$ remains true with dynamics

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A})$$

$$\Rightarrow \vec{\nabla}\times\left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0 \Rightarrow \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}V$$

$$\boxed{\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}}$$

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \Rightarrow \boxed{\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\rho/\varepsilon_0} \quad (*)$$

use $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = -\nabla^2 A + \vec{\nabla}(\vec{\nabla}\cdot\vec{A})$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t} \Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J} - \mu_0\varepsilon_0\vec{\nabla}\frac{\partial V}{\partial t} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}$$

$$\Rightarrow \boxed{\left(\nabla^2\vec{A} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right) - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu_0\varepsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\vec{J}} \quad (**)$$

Gauge Transformations – the V and $\vec{A}$ that give a particular $\vec{E}$ and $\vec{B}$ are not unique

Recall Statics $\quad \vec{E} = -\vec{\nabla}V \Rightarrow V' = V + C$ with C a constant gives the same $\vec{E}$

$$-\vec{\nabla}V' = -\vec{\nabla}V - \vec{\nabla}C = -\vec{\nabla}V + 0 = \vec{E}$$

$\vec{B} = -\vec{\nabla}\times\vec{A} \Rightarrow \vec{A}' = \vec{A} + \vec{\nabla}\lambda$ for any scalar function $\lambda(\vec{r})$ gives the same $\vec{B}$

$$\vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{\nabla}\lambda = \vec{\nabla}\times\vec{A} + 0 = \vec{B}$$

In dynamics $\vec{A}' = \vec{A} + \vec{\nabla}\lambda$ still gives the same $\vec{B}$ as does $\vec{A}$, just like statics

But now since $\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$, if we change from $\vec{A}$ to $\vec{A}'$ we would also change $\vec{E}$ unless we make some corresponding change in V

For $\vec{A} = \vec{A}' - \vec{\nabla}\lambda$,

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}V - \frac{\partial \vec{A}'}{\partial t} + \vec{\nabla}\frac{\partial \lambda}{\partial t}$$

$$= -\vec{\nabla}\left(V - \frac{\partial \lambda}{\partial t}\right) - \frac{\partial \vec{A}'}{\partial t}$$

So if we let $V' = V - \frac{\partial \lambda}{\partial t}$, then the change

$$\vec{A} \to \vec{A} + \vec{\nabla}\lambda = \vec{A}', \quad V \to V - \frac{\partial \lambda}{\partial t} = V'$$

leave the fields $\vec{E}$ and $\vec{B}$ unchanged.

The transformation $\vec{A}' = \vec{A} + \vec{\nabla}\lambda$, $V' = V - \frac{\partial \lambda}{\partial t}$ is called a gauge transformation

for any scalar function $\lambda(\vec{r},t)$, the gauge transformation leaves $\vec{B}$ and $\vec{E}$ unchanged

$$\vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times\vec{A}', \quad \vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}V' - \frac{\partial\vec{A}'}{\partial t}$$

We can therefore use the freedom, given by the arbitrary $\lambda(\vec{r},t)$, to impose some additional constraint on the potentials, in particular to make $\vec{\nabla}\cdot\vec{A}$ equal to some desired quantity which will simplify the equations (*) and (**). Making such a choice for $\vec{\nabla}\cdot\vec{A}$ is called "fixing the gauge".

1) Coulomb gauge : same as used in magnetostatics

require $\vec{\nabla}\cdot\vec{A} = 0$

How do we know we can always find such an $\vec{A}$? suppose we have an $\vec{A}$ such that $\vec{B} = \vec{\nabla}\times\vec{A}$ but $\vec{\nabla}\cdot\vec{A} \neq 0$. Then define $\vec{A}' = \vec{A} + \vec{\nabla}\lambda$. We want

$\vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\vec{A} + \nabla^2\lambda = 0$. This will be satisfied if we can find a $\lambda(\vec{r},t)$ that is a solution to

$$-\nabla^2\lambda = \underbrace{\vec{\nabla}\cdot\vec{A}}_{\text{this is a known function of } \vec{r},t \text{ since we know } \vec{A}}$$

This is just Poisson's equation!

For the case where sources are localized and our system is all of space out to infinity, the solution is

$$\lambda(\vec{r},t) = \frac{1}{4\pi}\int d^3r' \, \frac{\vec{\nabla}'\cdot\vec{A}(\vec{r}',t)}{|\vec{r}-\vec{r}'|}$$

So since we can always find the needed λ, we can always transform from $\vec{A}$ to a new $\vec{A}'$ with $\vec{\nabla}\cdot\vec{A}'=0$

In the Coulomb gauge with $\vec{\nabla}\cdot\vec{A}=0$

$(*) \Rightarrow \nabla^2 V = -\rho/\varepsilon_0$ Poisson's equation just like in statics!

Solution is $V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \, \frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}$

The above looks troubling because it says that V at position $\vec{r}$ at time t, is determined by all the charges at different $\vec{r}'$ at exactly the same time t! This implies action at a distance. The charge at $\vec{r}'$ effects the potential some distance away at $\vec{r}$ instantaneously. But we believe that information cannot travel instantaneously fast, it can travel no faster than the speed of light.

The solution to this paradox is to realize that in dynamics, $\vec{E}$ is not given solely by V, but rather $\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}$ depends also on $\vec{A}$

so the instantaneous relation between ρ and V can be canceled out by the term involving $\vec{A}$, and the relation between ρ and $\vec{E}$ will be causal (ie takes time for charge at $\vec{r}'$ to effect $\vec{E}$ at $\vec{r}$, and that transmission of effect travels with the speed of light)

since $\vec{\nabla}\cdot\vec{A}=0$

$$(**) \Rightarrow \left(\nabla^2 \vec{A} - \mu_0\varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}\right) = -\mu_0 \vec{j} + \mu_0\varepsilon_0 \vec{\nabla}\left(\frac{\partial V}{\partial t}\right)$$

use solution for V in terms of ρ

$$= -\mu_0 \vec{j} + \frac{\mu_0\varepsilon_0}{4\pi\varepsilon_0} \vec{\nabla}\int d^3r' \left(\frac{\partial \rho(\vec{r}',t)}{\partial t}\right)\frac{1}{|\vec{r}-\vec{r}'|}$$

use $\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\vec{j}$

$$= -\mu_0 \vec{j} + \frac{\mu_0}{4\pi}\vec{\nabla}\int d^3r' \frac{\left[-\vec{\nabla}'\cdot\vec{j}(\vec{r}',t)\right]}{|\vec{r}-\vec{r}'|}$$

relation between $\vec{A}$ and $\vec{j}$ is an integral-differential eqn

From Griffiths Appendix B

Corollary to Helmholtz Theorem:

any vector function $\vec{F}(\vec{r})$ that $\to 0$ sufficiently fast as $r \to \infty$ can be written as

$$(B.10)\quad \vec{F}(\vec{r}) = \underbrace{\vec{\nabla}\left(-\frac{1}{4\pi}\int d^3r' \frac{\vec{\nabla}'\cdot\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)}_{\vec{F}_L} + \underbrace{\vec{\nabla}\times\left(\frac{1}{4\pi}\int d^3r' \frac{\vec{\nabla}'\times\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)}_{\vec{F}_T}$$

$$= \vec{F}_L + \vec{F}_T$$

longitudinal part or curlfree part: $\vec{\nabla}\times\vec{F}_L = 0$

transverse part or divergenceless part: $\vec{\nabla}\cdot\vec{F}_T = 0$

can always write $\vec{F}_L = -\vec{\nabla}U$ $\qquad$ $\vec{F}_T = \vec{\nabla}\times\vec{W}$

compare to (**) in the Coulomb gauge
and we can write

$$\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} + \mu_0 \vec{J}_L$$

$$= -\mu_0 \vec{J}_T \qquad \text{since } \vec{J} = \vec{J}_L + \vec{J}_T$$

$$\boxed{\Box^2 \vec{A} = -\mu_0 \vec{J}_T}$$

vector potential determined by the transverse part of the current

$\Box^2 \equiv \nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}$

Note: $\vec{J}_T(\vec{r})$ is not locally related to $\vec{J}(\vec{r})$

For dynamic problems, a more convenient gauge to work in is the

2) Lorentz Gauge where we require

$$\vec{\nabla} \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} = 0$$

We can always find potentials $\vec{A}$ and V that satisfy this condition.

Proof: Suppose $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$ but

$\vec{\nabla} \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} \neq 0$. Then transform to

$$\vec{A}' = \vec{A} + \vec{\nabla} \lambda \quad , \quad V' = V - \frac{\partial \lambda}{\partial t}$$

Then

$$\vec{\nabla} \cdot \vec{A}' + \mu_0 \varepsilon_0 \frac{\partial V'}{\partial t} = \vec{\nabla} \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} + \nabla^2 \lambda - \mu_0 \varepsilon_0 \frac{\partial^2 \lambda}{\partial t^2}$$

we can make this vanish if we can find a $\lambda(\vec{r},t)$ that satisfies

$$\underbrace{-\nabla^2\lambda + \mu_0\varepsilon_0 \frac{\partial^2 \lambda}{\partial t^2}}_{-\Box^2\lambda} = \left[\vec{\nabla}\cdot\vec{A} + \mu_0\varepsilon_0 \frac{\partial V}{\partial t}\right]$$

known function of $\vec{r}, t$ since we know $\vec{A}$ ad V

just like Poissons equ $-\nabla^2\lambda = g$ always has a solution, one can show that the inhomogeneous wave equ $-\Box^2\lambda = g$ always has a solution

$\Rightarrow$ we can always find a $\lambda(\vec{r},t)$ that lets us transform to $\vec{A}'$ ad V' that are in the Lorentz gauge

In the Lorentz gauge $\vec{\nabla}\cdot\vec{A} = -\mu_0\varepsilon_0 \frac{\partial V}{\partial t}$
and so

$$(*) \Rightarrow \nabla^2 V - \mu_0\varepsilon_0 \frac{\partial^2 V}{\partial t^2} = -\rho/\varepsilon_0$$

$$(**) \Rightarrow \nabla^2\vec{A} - \mu_0\varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0\vec{J}$$

equations for V ad $\vec{A}$ now both have the same simple form

$$\Box^2 \equiv \nabla^2 - \mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2}$$

wave equation $\begin{cases} \Box^2 V = -\rho/\varepsilon_0 \\ \Box^2\vec{A} = -\mu_0\vec{J} \end{cases}$

is D'alembertain operator
wave equation operator
wave velocity $v^2 = \frac{1}{\mu_0\varepsilon_0}$

hence forth we will use the Lorentz gauge for all non-static problems.

Note: If we are in a static situation where

$$\frac{\partial V}{\partial t} = 0, \quad \frac{\partial \vec{A}}{\partial t} = 0, \quad \text{then}$$

$$\vec{\nabla}\cdot\vec{A} = -\mu_0 \varepsilon_0 \frac{\partial V}{\partial t} = 0 \quad \text{Lorentz gauge} \rightarrow \text{Coulomb gauge}$$

$$\Box^2 \rightarrow \nabla^2$$

so

$$\left.\begin{aligned} \Box^2 V &\rightarrow \nabla^2 V = -\rho/\varepsilon_0 \\ \Box^2 \vec{A} &\rightarrow \nabla^2 \vec{A} = -\mu_0 \vec{J} \end{aligned}\right\}$$ just the familiar Poisson equations found for statics in the Coulomb gauge

Lorentz Force

for a particle with charge q moving on trajectory $\vec{r}(t)$

$$\vec{F} = \frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v}\times\vec{B}) = q\left(-\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} + \vec{v}\times(\vec{\nabla}\times\vec{A})\right)$$

$$= -q\left(\vec{\nabla}V + \frac{\partial\vec{A}}{\partial t}\right) - q\left((\vec{v}\cdot\vec{\nabla})\vec{A} - \vec{\nabla}(\vec{v}\cdot\vec{A})\right)$$

$$= -q\left(\frac{\partial\vec{A}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{A} + \vec{\nabla}(V - \vec{v}\cdot\vec{A})\right)$$

$$\frac{\partial\vec{A}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{A} = \frac{d\vec{A}}{dt} \quad \text{convective derivative}$$

$$\frac{d}{dt}\left(\vec{A}(\vec{r}(t),t)\right) = \frac{\partial\vec{A}}{\partial t} + \frac{\partial\vec{A}}{\partial x}\frac{dx}{dt} + \frac{\partial\vec{A}}{\partial y}\frac{dy}{dt} + \frac{\partial\vec{A}}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial\vec{A}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{A}$$

change in $\vec{A}$ as seen by a particle moving with velocity $\vec{v}$

$$\Rightarrow \frac{d\vec{p}}{dt} = -q\left(\frac{d\vec{A}}{dt} + \vec{\nabla}(V - \vec{v}\cdot\vec{A})\right)$$

$$\frac{d}{dt}\underbrace{(\vec{p} + q\vec{A})}_{\text{"canonical" momentum } \vec{p}_{can}} = -\vec{\nabla}\underbrace{(qV - q\vec{v}\cdot\vec{A})}_{\text{"potential" } U}$$

$$\frac{d\vec{p}_{can}}{dt} = -\vec{\nabla}U$$

electromagnetic waves in a vacuum : for $\rho = \vec{j} = 0$

1) $\vec{\nabla}\cdot\vec{E} = 0$ 3) $\vec{\nabla}\cdot\vec{B} = 0$

2) $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ 4) $\vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$

$$\vec{\nabla}\times(2) \Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\underbrace{(\vec{\nabla}\cdot\vec{E})}_{=0 \text{ as } \rho=0} - \nabla^2\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$

from (4)

$$-\nabla^2\vec{E} = -\frac{\partial}{\partial t}\left(\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

$$\Rightarrow \left(\nabla^2 - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\right)\vec{E} = \Box^2\vec{E} = 0.$$

Similarly $\vec{\nabla}\times(4) \Rightarrow$
$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{B}}_{=0}) - \vec{\nabla}^2 B = \mu_0\varepsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

(2)

$$= \mu_0\varepsilon_0\frac{\partial}{\partial t}\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

$$\Rightarrow \left(\nabla^2 - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\right)\vec{B} = \Box^2\vec{B} = 0.$$

for any function $f(\vec{r},t)$,

$$\nabla^2 f - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$$

is "wave equation"
describes waves moving with speed v.

$\Rightarrow$ Maxwell's eqn's in vacuum have wave solutions that move with speed $= \frac{1}{\sqrt{\varepsilon_0\mu_0}} = 3\times10^8$ m/s.

This combination of $\varepsilon_0\mu_0$ turns out to be exactly the speed of light! This realization of Maxwell's demonstrated that light was just an electro-magnetic wave!