Inhomogeneous wave equation

$$\Box^2 f(\vec{r},t) = g(\vec{r},t) \quad \text{where } g \text{ is a given source function}$$

$$f(\vec{r},t) = \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \; \tilde{f}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$g(\vec{r},t) = \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \; \tilde{g}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

substitute in

$$\Rightarrow \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \; \tilde{f}(\vec{k},\omega) \left[-k^2 + \frac{\omega^2}{v^2}\right] e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \; \tilde{g}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

equate Fourier components

$$\Rightarrow \tilde{f}(\vec{k},\omega)\left[-k^2 + \frac{\omega^2}{v^2}\right] = \tilde{g}(\vec{k},\omega)$$

$$\Rightarrow f(\vec{k},\omega) = \frac{\tilde{g}(\vec{k},\omega)}{\frac{\omega^2}{v^2} - k^2}$$

$$f(\vec{r},t) = \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \; \frac{\tilde{g}(\vec{k},\omega)}{\frac{\omega^2}{v^2} - k^2}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \; \frac{e^{i(\vec{k}\cdot\vec{r}-\omega t)}}{\frac{\omega^2}{v^2} - k^2} \cdot \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt' \; g(\vec{r}',t')\, e^{-i(\vec{k}\cdot\vec{r}'-\omega t')}$$

$$f(\vec{r},t) = \int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt' \, g(\vec{r}',t') \boxed{\int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \frac{1}{(2\pi)^4} \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} e^{-i\omega(t-t')}}{\left(\frac{\omega^2}{v^2} - k^2\right)}}$$

Green's function for wave eqn
$G(\vec{r}-\vec{r}', t-t')$

$$f(\vec{r},t) = \int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt' \, G(\vec{r}-\vec{r}', t-t') \, g(\vec{r}',t')$$

Same form as solution to Poisson's Eqn

$$-\nabla^2 V = \rho/\varepsilon_0 \Rightarrow V(\vec{r}) = \int d^3r \, \frac{\rho(\vec{r}')}{\varepsilon_0} \, G(\vec{r},\vec{r}')$$

$$G(\vec{r},\vec{r}') = \frac{1}{4\pi} \quad \frac{1}{|\vec{r}-\vec{r}'|}$$

Green's function for Poisson's eqn

Polarization

vector wave $\quad \vec{f}(\vec{r},t) = \vec{A}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

where $\vec{A} = A\hat{m}$, A may be complex number to include phase factor.

if $\hat{m}$ is $\parallel \hat{k}$ we have longitudinal polarization

" $\hat{m}$ is $\perp \hat{k}$ " " transverse polarization

$\cancel{\vec{\nabla}\cdot\vec{E}=0} \quad \cancel{\vec{\nabla}\cdot\vec{B}=0}$

$\cancel{\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}} \quad \cancel{\vec{\nabla}\times\vec{B}=\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}}$

Circular polarization: sum of orthogonal transverse polarizations, $\pi/2$ out of phase

Consider $\quad \vec{f}(\vec{r},t) = A\hat{m}_1 e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \underbrace{iA\hat{m}_2 e^{i(\vec{k}\cdot\vec{r}-\omega t)}}_{A\hat{m}_2 e^{i(\vec{k}\cdot\vec{r}-\omega t+\frac{\pi}{2})}}$

where $\hat{m}_1 \perp \hat{m}_2 \perp \hat{k}$ is right handed coord system

$$\vec{f} = A(\hat{m}_1 + i\hat{m}_2)e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

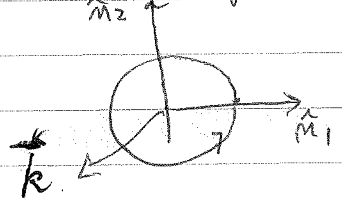
to get physical field, take Real part of complex form

$$\Rightarrow \vec{f} = A\hat{m}_1 \cos(\vec{k}\cdot\vec{r}-\omega t) - A\hat{m}_2 \sin(\vec{k}\cdot\vec{r}-\omega t)$$

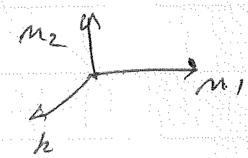
$$= A\hat{m}_1 \cos(\omega t - \vec{k}\cdot\vec{r}) + A\hat{m}_2 \sin(\omega t - \vec{k}\cdot\vec{r})$$

$$|\vec{f}|^2 = A^2\cos^2(\vec{k}\cdot\vec{r}-\omega t) + A^2\sin^2(\vec{k}\cdot\vec{r}-\omega t) = A^2$$

amplitude constant, but direction of $\vec{f}$ rotates in time counter clockwise with frequency ω.



$\vec{f} = A(\hat{n}_1 + i\,\hat{n}_2)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$, where $\hat{n}_1, \hat{n}_2, \hat{k}$ form right handed coord system



is a "Right handed" <u>circularly polarized</u> wave

$\vec{f} = A(\hat{n}_1 - i\,\hat{n}_2)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ is a "Left handed" circularly polarized wave - direction of $\vec{f}$ rotates in time clockwise

<u>Plane EM waves in a vacuum</u>

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

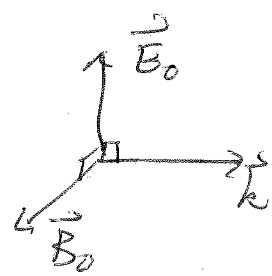
Assume solutions of form $\left.\begin{aligned}\vec{E} &= \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\\ \vec{B} &= \vec{B}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\end{aligned}\right\}$ where $\omega = \frac{1}{\sqrt{\mu_0\varepsilon_0}}k = ck$

Maxwell's eqn's becom

1) $i\vec{k}\cdot\vec{E}_0 = 0$ 3) $i\vec{k}\cdot\vec{B}_0 = 0$

2) $i\vec{k}\times\vec{E}_0 = +i\omega\vec{B}_0$ 4) $i\vec{k}\times\vec{B}_0 = \mu_0\varepsilon_0(-i\omega)\vec{E}_0$

(1) and (3) $\Rightarrow$ EM waves are transverse polarized.
$\vec{E}_0$ and $\vec{B}_0$ both $\perp$ to $\vec{k}$.

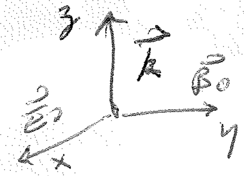


2) $\Rightarrow \vec{B}_0 = \frac{\vec{k}}{\omega}\times\vec{E}_0 = \frac{1}{c}\hat{k}\times\vec{E}_0 \Rightarrow \vec{B}_0 \perp \vec{E}_0$

$|\vec{B}_0| = \frac{1}{c}|\vec{E}_0|$

↑ very important factor $\frac{1}{c}$!

Since Lorentz force is $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$, the force on a charged particle due to an electromagnetic wave is predominantly from the electric field $\vec{E}$. The force due to the magnetic field is $\sim v B_0 = \left(\frac{v}{c}\right)E_0$, is reduced by a factor $\left(\frac{v}{c}\right) \ll 1$, unless charge is moving relativistically fast.



$\vec{k} = k\hat{z}$

energy + momentum in EM wave:

$\vec{E}(r,t) = E_0 \cos(kz - \omega t)\,\hat{x}$ $\qquad \vec{B}(r,t) = \frac{1}{c} E_0 \cos(kz - \omega t)\,\hat{y}$

energy density

$$u_{EB} = \frac{\varepsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 = \frac{\varepsilon_0}{2} E_0^2 \cos^2(kz - \omega t) + \frac{1}{2\mu_0 c^2} E_0^2 \cos^2(kz - \omega t)$$

$$= \frac{1}{2} E_0^2 \cos^2(kz - \omega t) \underbrace{\left[\varepsilon_0 + \frac{1}{\mu_0 c^2} \right]}_{\underbrace{\varepsilon_0 + \frac{\mu_0 \varepsilon_0}{\mu_0}}_{2\varepsilon_0}} \qquad \text{use } c^2 = \frac{1}{\mu_0 \varepsilon_0}$$

$$u_{EB} = \varepsilon_0 E_0^2 \cos^2(kz - \omega t)$$

Note: when taking the product of 2 factors of $\vec{E}$ or $\vec{B}$, important to take Re parts first, if using complex notation

energy current

$$\vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B})$$

$$= \frac{1}{\mu_0 c} E_0^2 \cos^2(kz - \omega t)(\hat{x} \times \hat{y}) = c\,\varepsilon_0 E_0^2 \cos^2(kz - \omega t)\,\hat{z}$$

using $\frac{1}{\mu_0 c} = \frac{c}{\mu_0 c^2} = \frac{c\,\mu_0 \varepsilon_0}{\mu_0} = c\,\varepsilon_0$

$$\vec{S} = c\,u_{EB}\,\hat{z}$$

momentum density $\quad \vec{p}_{EB} = \frac{1}{c^2}\vec{S} = \frac{u_{EB}}{c}\,\hat{z}$

$\Rightarrow \quad u_{EB} = c\,|\vec{p}_{EB}|$ — energy-momentum relation of photons

Since for visible light, $\lambda \sim 5 \times 10^{-7}$ m ~ 5000 Å

$$T = \frac{\lambda}{c} = \frac{5 \times 10^{-7}}{3 \times 10^{8}} \text{ sec} = 1.6 \times 10^{-15} \text{ sec}$$

for most classical measurements, on macroscopic scale, the measurement will average over many oscillations of the wave. Therefore one is interested in averages

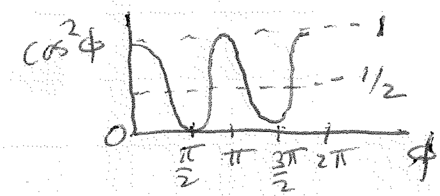
$$\langle u_{EB} \rangle = \frac{1}{T} \int_0^T dt \; u_{EB}$$ average over one period of oscillation

$$= \frac{\lambda}{c} \int_0^T dt \; \varepsilon_0 E_0^2 \cos^2(kz - \omega t)$$

$T = \frac{2\pi}{\omega}$ is period of oscillation $= \frac{\lambda}{c}$

$$\langle u_{EB} \rangle = \frac{1}{2} \varepsilon_0 E_0^2$$ average of $\cos^2(\phi)$ over one period is $\frac{1}{2}$

$$\langle \vec{S} \rangle = c \langle u_{EB} \rangle \hat{z}$$

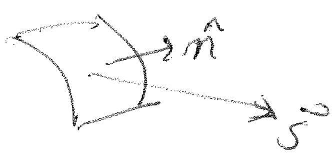


$$\langle \vec{p}_{EB} \rangle = \frac{1}{c} \langle u_{EB} \rangle \hat{z}$$

<u>intensity</u> = average (over time) power per area transported by wave

intensity $\quad I = |\langle \vec{S} \rangle|$ magnitude of energy current $\sim$ (amplitude of field)^2

$\langle \vec{S} \rangle \cdot \hat{n}$ = average power per area transported through surface with normal $\hat{n}$



Maxwell's Eqn in Matter

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho_{tot} \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}_{tot} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

want to write $\rho_{tot} = \rho_{free} + \rho_b \qquad \vec{J}_{tot} = \vec{J}_{free} + \vec{J}_b$

in statics : $\rho_b = -\vec{\nabla}\cdot\vec{P} \qquad \vec{J}_b = \vec{\nabla}\times\vec{M}$

in dynamics : conservation of bound charge $\Rightarrow \vec{\nabla}\cdot\vec{J}_b = -\frac{\partial\rho_b}{\partial t}$

$$\underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{M})}_{=\,0} = +\frac{\partial}{\partial t}\underbrace{(\vec{\nabla}\cdot\vec{P})}_{\neq\,0}$$

something must be missing! The bound current arising from $\vec{M}$ must not be all the bound current. There must be bound current arising from a time varying $\vec{P}$.

bound current from polarization, $\vec{J}_p$ must satisfy

$$\vec{\nabla}\cdot\vec{J}_p = -\frac{\partial\rho_b}{\partial t} = \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{P} = \nabla\cdot\left(\frac{\partial\vec{P}}{\partial t}\right)$$

$$\Rightarrow \vec{J}_p = \frac{\partial\vec{P}}{\partial t}$$

$$\Rightarrow \vec{J}_b = \vec{\nabla}\times\vec{M} + \frac{\partial\vec{P}}{\partial t}$$