Do the fields $\vec{E}$ and $\vec{B}$ inside such a material with an $\varepsilon(\omega)$ obey the wave equation when $\rho_f = 0$, $\vec{J}_f = 0$?

For simplicity we take a material with no magnetic response

$$\mu = \mu_0 \quad , \quad \vec{H} = \frac{\vec{B}}{\mu_0}$$

Ampere's law when $\vec{J}_f = 0$ is

$$\vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t} \rightarrow \vec{\nabla}\times\vec{B} = \mu_0 \frac{\partial\vec{D}}{\partial t}$$

use $\vec{D}(\vec{r},t) = \int_{-\infty}^{\infty}\frac{dt'}{2\pi} E(r,t')\,\tilde{\varepsilon}(t-t')$ where $\tilde{\varepsilon}(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\,\varepsilon(\omega)$ is F.T. of $\varepsilon(\omega)$

Then

$$\vec{\nabla}\times\vec{B} = \mu_0 \frac{\partial}{\partial t}\int\frac{dt'}{2\pi}\vec{E}(r,t')\,\tilde{\varepsilon}(t-t') = \mu_0\int\frac{dt'}{2\pi}\vec{E}(\vec{r},t')\,\frac{\partial\tilde{\varepsilon}(t-t')}{\partial t}$$

Take curl of both sides

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = -\nabla^2\vec{B} = \mu_0\int\frac{dt'}{2\pi}\left[\vec{\nabla}\times\vec{E}(\vec{r},t')\right]\frac{\partial\tilde{\varepsilon}(t-t')}{\partial t}$$

use $\vec{\nabla}\times\vec{E}(\vec{r},t') = -\frac{\partial\vec{B}(r,t')}{\partial t'}$ Faraday's law

$$-\nabla^2\vec{B}(\vec{r},t) = -\mu_0\int\frac{dt'}{2\pi}\frac{\partial\vec{B}(\vec{r},t')}{\partial t'}\frac{\partial\tilde{\varepsilon}(t-t')}{\partial t}$$

use $\frac{\partial\tilde{\varepsilon}(t-t')}{\partial t} = -\frac{\partial\tilde{\varepsilon}(t-t')}{\partial t'}$

$$-\nabla^2\vec{B}(\vec{r},t) = \mu_0\int_{-\infty}^{\infty}\frac{dt'}{2\pi}\frac{\partial\vec{B}(\vec{r},t')}{\partial t'}\frac{\partial\tilde{\varepsilon}(t-t')}{\partial t'}$$

integrate by parts

$$-\nabla^2\vec{B}(\vec{r},t) = -\mu_0\int\frac{dt'}{2\pi}\frac{\partial^2\vec{B}(\vec{r},t')}{\partial t'^2}\tilde{\varepsilon}(t-t')$$

$$\nabla^2 \vec{B}(\vec{r},t) - \mu_0 \int \frac{dt'}{2\pi} \frac{\partial^2 B(\vec{r},t')}{\partial t'^2} \tilde{\varepsilon}(t-t') = 0$$

looks somewhat like the wave equation, except for the integration.

Suppose $\tilde{\varepsilon}(t-t') = 2\pi\varepsilon\,\delta(t-t')$ with ε a constant

Then we would get

$$\nabla^2 \vec{B}(\vec{r},t) - \mu_0 \varepsilon \int_{-\infty}^{\infty} dt' \frac{\partial^2 \vec{B}(\vec{r},t')}{\partial t'^2} \delta(t-t') = 0$$

$$\Rightarrow \nabla^2 B(\vec{r},t) - \mu_0 \varepsilon \frac{\partial^2 \vec{B}(\vec{r},t)}{\partial t^2} = 0$$ The wave equation!

So only if the responce function $\tilde{\varepsilon}(t) = 2\pi\varepsilon\,\delta(t)$, ie it is instantaneous in time, will $\vec{B}$ solve the wave equation with wave velocity $v = \frac{1}{\sqrt{\mu_0 \varepsilon}}$

What $\varepsilon(\omega)$ corresponds to $\tilde{\varepsilon}(t-t') = 2\pi\varepsilon\,\delta(t-t')$?

$$\varepsilon(\omega) \equiv \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{i\omega t} \tilde{\varepsilon}(t)$$ definition of Fourier transform

$$= \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{-i\omega t} 2\pi\varepsilon\,\delta(t) = \varepsilon$$

So only when $\varepsilon(\omega) = \varepsilon$ is a constant indep of ω, will the fields obey the wave equation!

$\Longrightarrow$ Maxwell's Equ only look simple when expressed in terms of <u>Fourier Transforms</u>.

For pure sinusoidal solutions: $\vec{E}(\vec{r},t) = \vec{E}_\omega\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$\vec{B}(\vec{r},t) = \vec{B}_\omega\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$\vec{H}(\vec{r},t) = \vec{H}_\omega\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$\vec{D}(\vec{r},t) = \vec{D}_\omega\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

for EM waves in dielectric, assume $\rho_f = \vec{J}_f = 0$

Maxwell's Equs: $\vec{\nabla}\cdot\vec{D}=0$, $\vec{\nabla}\cdot\vec{B}=0$, $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$, $\vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t}$

assume $\mu=\mu_0 \Rightarrow \vec{H}_\omega = \frac{1}{\mu_0}\vec{B}_\omega$

dielectric response given by $\varepsilon(\omega) \Rightarrow \vec{D}_\omega = \varepsilon(\omega)\vec{E}_\omega$

<u>For $\rho_f = \vec{J}_f = 0$</u>, Maxwell's Equs in terms of the Fourier amplitudes are then

1) $i\vec{k}\cdot\vec{D}_\omega = i\varepsilon(\omega)\,\vec{k}\cdot\vec{E}_\omega = 0 \quad\Rightarrow \vec{k}\cdot\vec{E}_\omega = 0 \quad \vec{k}\perp\vec{E}_\omega$

2) $i\vec{k}\cdot\vec{B}_\omega = 0 \quad\Rightarrow \vec{k}\cdot\vec{B}_\omega = 0 \quad \vec{k}\perp\vec{B}_\omega$

$\left.\begin{matrix}\vec{k}\perp\vec{E}_\omega \\ \vec{k}\perp\vec{B}_\omega\end{matrix}\right\}$ transverse

3) Faraday $\quad i\vec{k}\times\vec{E}_\omega = i\omega\vec{B}_\omega$

4) Ampere $\quad i\vec{k}\times\vec{H}_\omega = -i\omega\vec{D}_\omega \Rightarrow \frac{i\vec{k}}{\mu_0}\times\vec{B}_\omega = -i\omega\varepsilon(\omega)\vec{E}_\omega$

$k\times(\text{Faraday}) = i\vec{k}\times(\vec{k}\times\vec{E}_\omega) = i\omega(\vec{k}\times\vec{B}_\omega)$ substitute in from Ampere

$= -i\omega^2\varepsilon(\omega)\mu_0\vec{E}_\omega$

$$\vec{k}\times(\vec{k}\times\vec{E}_\omega) = \vec{k}\,\underbrace{(\vec{k}\cdot\vec{E}_\omega)}_{=0 \text{ by } (1)} - \vec{E}_\omega(\vec{k}\cdot\vec{k}) = -\omega^2\varepsilon(\omega)\mu_0\vec{E}_\omega$$

$$\Rightarrow k^2\vec{E}_\omega = \omega^2\varepsilon(\omega)\mu_0\vec{E}_\omega$$

$$\Rightarrow \boxed{\begin{aligned} k^2 &= \omega^2 \varepsilon(\omega)\mu_0 \\ k^2 &= \frac{\omega^2}{c^2}\left(\frac{\varepsilon(\omega)}{\varepsilon_0}\right)\end{aligned}}$$

use $\frac{1}{c^2} = \mu_0 \varepsilon_0$

"dispersion" relation for waves in dielectric

dispersion relation determines wave vector k, for a given frequency ω.

Note $\frac{\omega^2}{k^2} \neq$ constant $\Rightarrow \vec{E}$ is <u>not</u> solution of a wave equation $\Box^2 \vec{E} = 0$.
different frequencies travel with <u>different</u> speeds.

Since $\varepsilon(\omega)$ is complex $\quad \varepsilon(\omega) = \underbrace{\varepsilon_1(\omega)}_{Re[\varepsilon]} + i \underbrace{\varepsilon_2(\omega)}_{Im[\varepsilon]}$

then in general the wavevector is also complex

$$k = k_1 + i k_2 = \pm \frac{\omega}{c}\sqrt{\frac{\varepsilon_1}{\varepsilon_0} + i\frac{\varepsilon_2}{\varepsilon_0}}$$

For a wave traveling in $\hat{z}$ direction, $\vec{k} = k\hat{z}$, we have

$$\vec{E}(\vec{r},t) = \vec{E}_\omega e^{i(\vec{k}\cdot\vec{r} - \omega t)} = \vec{E}_\omega e^{i((k_1 + i k_2)z - \omega t)}$$

$$= \vec{E}_\omega e^{-k_2 z} e^{i(k_1 z - \omega t)}$$

If choose $+\sqrt{\ }$ solution for k_1, so that wave propagates in $+\hat{z}$ direction, then should take $+\sqrt{\ }$ solution for k_2, so that wave decays as it propagates into material decay length $= 1/k_2$ k_2 is called the attenuation

Since intensity is $\sim E^2$ decays as $e^{-2k_2 z}$, $2k_2$ is called the absorption coefficient

physical origin of decay: EM wave excits atom to oscillate. Oscillations pump energy into other degrees of freedom, due to damping γ. $\Rightarrow$ EM wave is pumping energy into material $\Rightarrow$ energy contained in EM wave should decrease as it propagates into material $\Rightarrow$ amplitude decays.

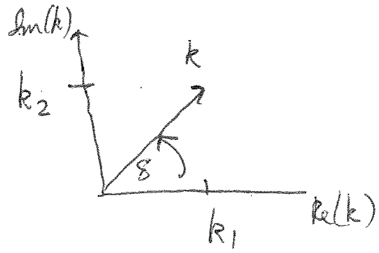
phase velocity of wave $\quad v_p \equiv \dfrac{\omega}{k_1}$ depends on frequency

index of refraction $\quad n \equiv \dfrac{c}{v_p} = \dfrac{ck_1}{\omega}$ depends on freq

Lets look now at magnetic field. From Faraday

$$\vec{B}_\omega = \frac{\vec{k}}{\omega} \times \vec{E}_\omega = \frac{(k_1 + ik_2)}{\omega}\,\hat{z} \times \vec{E}_\omega$$

write $k_1 + ik_2 = \sqrt{k_1^2 + k_2^2}\, e^{i\delta} = |k| e^{i\delta}$ where $\delta = \arctan\left(\dfrac{k_2}{k_1}\right)$ is phase of k



$$\vec{B}_\omega = \frac{|k|}{\omega}\,\hat{z} \times \vec{E}_\omega\, e^{i\delta}$$

$$\vec{B}(\vec{r},t) = \frac{|k|}{\omega}(\hat{z}\times\vec{E}_\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t+\delta)}$$

$$= \frac{|k|}{\omega}(\hat{z}\times\vec{E}_\omega)\, e^{-k_2 z}\, e^{i(k_1 z-\omega t+\delta)}$$

Physical fields:

$$\vec{E}(\vec{r},t) = \vec{E}_\omega e^{-k_2 z}\cos(k_1 z-\omega t)$$

$$\vec{B}(\vec{r},t) = (\hat{z}\times\vec{E}_\omega)\frac{|k|}{\omega} e^{-k_2 z}\cos(k_1 z-\omega t+\delta)$$

$\Rightarrow$ (1) $\vec{E}$ and $\vec{B}$ are transverse to $\vec{k}$, and $\vec{E}\perp\vec{B}$

(2) ratio of amplitudes $\frac{|\vec{B}|}{|\vec{E}|} = \frac{|k|}{\omega} = \frac{\sqrt{k_1^2+k_2^2}}{\omega} = \sqrt{\left|\frac{\varepsilon(\omega)}{\varepsilon_0}\right|}\,\frac{1}{c}$

(3) $\vec{B}$ wave is shifted with respect to $\vec{E}$ wave by phase shift $\delta = \arctan(k_2/k_1)$ (see Fig 8.21 in text)

Summary

Main consequenses of complex $\varepsilon(\omega)$

1) Waves decay as they propagate $\sim e^{-k_2 z}$

2) $\vec{E}$ and $\vec{B}$ waves shifted in phase by $\delta = \arctan(k_2/k_1)$

If $\varepsilon_2 = \mathrm{Im}[\varepsilon(\omega)] = 0$, then $\underbrace{\varepsilon \text{ real}, \Rightarrow k \text{ real}}_{\text{and if } \varepsilon_1>0}$, $k_2=0$

$\Rightarrow$ no decay and no phase shift.

Main consequenses of freq dependent $\varepsilon(\omega)$

(1) $\vec{E}(t)$ and $\vec{D}(t)$ non-locally related in time

(2) waves of different ω travel with different velocities $v_p = \frac{\omega}{k_1}$

(3) dispersion – wave pulses do not travel with v_p, and do not keep their shape as they propagate

Phase velocity and group velocity and dispersion

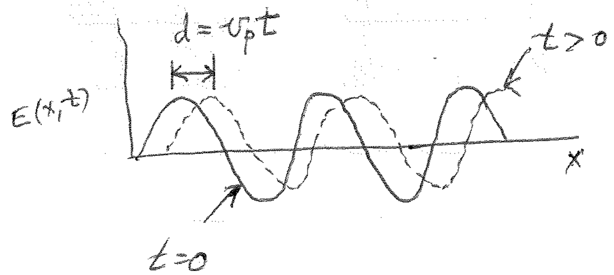
$$k^2 = \frac{\omega^2}{c^2}\frac{\varepsilon(\omega)}{\varepsilon_0}$$ For simplicity, assume $\varepsilon(\omega)$ is real and positive

$$k = \frac{\omega}{c}\sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}} \qquad v_p = \frac{\omega}{k} = c\sqrt{\frac{\varepsilon_0}{\varepsilon(\omega)}} = \frac{c}{n}$$

index of refraction $n(\omega) = \sqrt{\dfrac{\varepsilon(\omega)}{\varepsilon_0}} = \sqrt{K(\omega)}$ dielectric function

sinusoidal waves $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ propagate with different phase speeds $v_p(\omega)$ for different ω.

v_p is speed with which peaks in oscillation move to right



for $\vec{E} = E e^{i(kx-\omega t)}$ with $\omega = v_p(\omega)\,k$

If take linear superposition of many sinusoidal waves, then each different freq ω, moves with different speed $v_p(\omega)$. So the shape of the wave is <u>not</u> preserved in time.
[This is another way to see that waves in a dielectric do not solve the wave equation - for the wave equation, all freq move with same speed v indep of ω, and the shape of the wave is always preserved in time, i.e. solutions are always of form $g(\vec{k}\cdot\vec{r}-\omega t)$]

Consider a superposition of waves all traveling in $\hat{z}$ direction

$$\vec{E}(\vec{r},t) = \int_{-\infty}^{\infty} d\omega\, \vec{E}_\omega\, e^{i(k(\omega)z - \omega t)} \qquad k(\omega) = \frac{\omega}{c}\sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}}$$

At $\vec{r}=0$, $\vec{E}(0,t) = \int_{-\infty}^{\infty} d\omega\, \vec{E}_\omega\, e^{-i\omega t}$ so $\vec{E}_\omega$ is F.T. of $\vec{E}(0,t)$

At some position $\vec{r} \neq 0$

$$\vec{E}(\vec{r},t) = \int_{-\infty}^{\infty} d\omega\, \vec{E}_\omega\, e^{i(k(\omega)z - \omega t)}$$

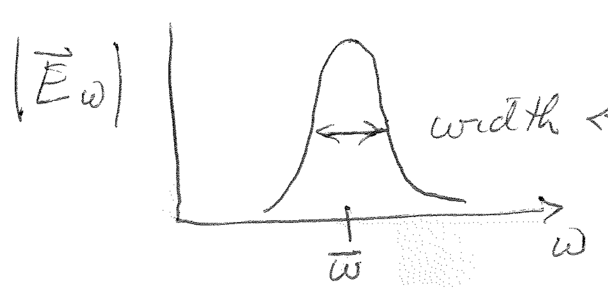
if no dispersion, ie $k = \frac{\omega}{c}\sqrt{\frac{\varepsilon}{\varepsilon_0}} = \frac{\omega}{v_p}$ with v_p indep of ω

Then $\vec{E}(\vec{r},t) = \int_{-\infty}^{\infty} d\omega\, \vec{E}_\omega\, e^{-i\omega(t - z/v_p)}$

$= \vec{E}(0, t - z/v_p) \leftarrow$ form of solution to wave equation

field at z at time t, is same field as was at $z=0$ at the earlier time $t - z/v_p \Rightarrow$ wave moved distance z in time $z/v_p \Rightarrow$ speed of wave is v_p

Suppose now that $\varepsilon(\omega)$ does depend on ω, so there is dispersion. Suppose $\vec{E}_\omega$ is strongly peaked about some average $\bar{\omega}$



then $k(\omega) \cong k(\bar{\omega}) + \left.\frac{dk}{d\omega}\right|_{\bar{\omega}} (\omega - \bar{\omega}) + \cdots$

$$\vec{E}(\vec{r},t) = \int d\omega\, \vec{E}_\omega\, e^{i(k(\bar{\omega})z + \frac{dk}{d\omega}\omega z - \frac{dk}{d\omega}\bar{\omega} z - \omega t)}$$

$$= e^{i(k(\bar{\omega}) - \frac{dk}{d\omega}\bar{\omega})z} \int_{-\infty}^{\infty} d\omega\, \vec{E}_\omega\, e^{-i\omega(t - \frac{dk}{d\omega} z)}$$

$$= \underbrace{e^{i(k(\bar{\omega}) - \frac{dk}{d\omega}\bar{\omega})z}}_{\text{phase factor}} \; \underbrace{\vec{E}(0, t - \tfrac{dk}{d\omega} z)}_{\text{envelope - determines shape of pulse}}$$

intensity of wave $\sim |\vec{E}|^2$

$$|\vec{E}|^2(\vec{r},t) = |\vec{E}|^2(0, t - \tfrac{dk}{d\omega} z)$$

intensity travels with velocity $v_g = 1/\left(\frac{dk}{d\omega}\right)_{\bar{\omega}} = \frac{d\omega}{dk}$ = <u>group velocity</u>

<u>not</u> with average phase velocity $\bar{v}_p = \frac{\bar{\omega}}{k(\bar{\omega})}$

only when $\varepsilon(\omega)$ is indep of ω will $v_p = v_g$

$$\frac{1}{v_g} = \frac{dk}{d\omega} = \frac{d}{d\omega}\left[\frac{\omega}{c} n(\omega)\right] = \frac{n}{c} + \frac{\omega}{c}\frac{dn}{d\omega} = \frac{1}{v_p} + \frac{\omega}{c}\frac{dn}{d\omega}$$

$$v_g = \frac{v_p}{1 + \frac{v_p}{c}\omega\frac{dn}{d\omega}} \Rightarrow \text{when } \frac{dn}{d\omega} > 0,\; v_g < v_p \quad ①$$

$$\text{when } \frac{dn}{d\omega} < 0,\; v_g > v_p \quad ②$$

case ① is called "normal" dispersion
case ② is called "anomalous" dispersion

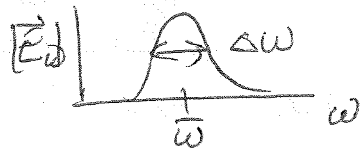
Our result $\vec{E}^2(r,t) = \vec{E}^2(0, t - \frac{dk}{d\omega} z)$ looks like we still preserve shape of wave – but this is due to the simplicity of our approximation. If we kept to next order, i.e. used $k(\omega) = k(\bar{\omega}) + \frac{dk}{d\omega}(\omega - \bar{\omega})$

$$+ \frac{1}{2}\frac{d^2k}{d\omega^2}(\omega-\bar{\omega})^2$$

one would find that the wave pulse changes shape as it propagates – in particular, it spreads.

A simple way to estimate this effect:

If pulse initially has width $\Delta\omega$ about $\bar{\omega}$, i.e. $\vec{E}_\omega$ looks like



there is a spread in group velocities

$$\Delta v_g \simeq \left|\frac{dv_g}{d\omega}\right| \Delta\omega = \left|\frac{d}{d\omega}\left(\frac{1}{dk/d\omega}\right)\right| \Delta\omega$$

$$= \frac{1}{(dk/d\omega)^2}\left|\frac{d^2k}{d\omega^2}\right|\Delta\omega = v_g^2 \left|\frac{d^2k}{d\omega^2}\right| \Delta\omega$$

So if pulse take a time $T = z/v_g$ to reach point z from the origin, there is also a spread in arrival times

$$\Delta T = \Delta(z/v_g) = \frac{z}{v_g^2}\Delta v_g = z\left|\frac{d^2k}{d\omega^2}\right|\Delta\omega$$

ΔT gives a spreading of width of the wave pulse, that grows linearly with the distance z traveled.

For a pulse of width $\Delta\omega$, the width in time is

$$\Delta t \sim \frac{1}{\Delta\omega}$$ (like uncertainty principle in QM)

$$\Rightarrow \Delta T \simeq z \left|\frac{d^2k}{d\omega^2}\right| \frac{1}{\Delta t}$$

$\Rightarrow$ the sharper the pulse is initially, (ie the smaller Δt) the faster it spreads as it travels (ie the larger ΔT is).