

## Abraham - Lorentz Equation

We saw in the hydrogen example, how one can treat radiation-reaction force as a small perturbation of the charge's motion when  $E_{\text{rad}} \ll E_0$  — the electron remained in a circular orbit whose radius decreased slowly compared to period of revolution  $1/\omega_0$ , as the energy is radiated away.

Now we want a more general equation of motion, including the radiation-reaction force, that will be valid even on time scales  $t \sim \tau$ , i.e. when radiation reaction is not a small perturbation.

### Energy conservation

Newton's eqn  $m\dot{\vec{v}} = \vec{F}_{\text{ext}}$  ← externally applied force

want to generalize to  $m\dot{\vec{v}} = \vec{F}_{\text{ext}} + \vec{F}_{\text{rad}}$   
↑ radiation-reaction force

We saw that power radiated away by moving charges

$$\text{is } P(t) = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2}{c^3} |\ddot{\vec{a}}(t)|^2$$

We therefore would expect that

$$\vec{F}_{\text{rad}} \cdot \vec{v} = \left( \vec{F}_{\text{rad}} \cdot \frac{d\vec{r}}{dt} \right)_{\text{loss}} = -P(t)$$

work done by  $\vec{F}_{\text{rad}}$   
per unit time  
= - power charge  
loses

Not strictly true -  $P(t)$  has in it only the power that is radiated away - it does not include the energy flux from the "near zone" or "velocity" fields, which also can instantaneously affect charges motion. But in HW problem we saw that for a periodic motion, the ~~the~~ contribution of the "near zone" fields to the energy flux average to zero. So if we integrate over one cycle of the charges motion

$$\int_0^T dt \vec{F}_{\text{rad}} \cdot \vec{v} = -\frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2}{c^3} \int_0^T dt \left( \frac{d\vec{v}}{dt} \right)^2$$

$$= \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2}{c^3} \int_0^T dt \frac{d\vec{v}}{dt} \cdot \frac{d\vec{v}}{dt}$$

integrate by parts

$$= \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2}{c^3} \left[ \int_0^T dt \frac{d^2\vec{v}}{dt^2} \cdot \vec{v} - \left[ \frac{d\vec{v}}{dt} \cdot \vec{v} \right]_0^T \right]$$

↑  
 vanishes, since by definition of cycle period  $T$ ,  $\vec{v}(T) = \vec{v}(0)$   
 $\dot{\vec{v}}(T) = \dot{\vec{v}}(0)$

$$\Rightarrow \int_0^T dt \left[ \vec{F}_{\text{rad}} - \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2}{c^3} \ddot{\vec{a}} \right] \cdot \vec{v} = 0$$

Therefore a reasonable guess is

$$\vec{F}_{\text{rad}} = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2}{c^3} \ddot{\vec{a}}$$

= 0 unless acceleration is changing

Defining time constant  $\tau = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2}{c^3 m}$

$$\vec{F}_{\text{rad}} = m\tau \dot{\vec{a}}$$

Abraham Lorentz equation:

$$\begin{aligned} m\ddot{\vec{a}} &= \vec{F}_{\text{ext}} + \vec{F}_{\text{rad}} \\ &= \vec{F}_{\text{ext}} + m\tau \dot{\vec{a}} \end{aligned}$$

$$m(\ddot{\vec{a}} - \tau \dot{\vec{a}}) = \vec{F}_{\text{ext}} \quad \left. \begin{array}{l} \text{includes time averaged} \\ \text{effects of radiation-reaction} \end{array} \right\}$$

3<sup>rd</sup> order in time  $\Rightarrow$  to specify a unique solution one needs to know  $\vec{r}(t=0)$ ,  $\vec{v}(t=0)$ , and  $\dot{\vec{a}}(t=0)$ ! very different from Newtonian mechanics where all you need to specify is  $\vec{r}(t=0)$  and  $\vec{v}(t=0)$  to get unique solution.

For example: if  $\vec{F}_{\text{ext}} = 0$ ,  $\vec{r}(0) = 0$ ,  $\vec{v}(0) = 0$ , Abraham Lorentz equation still gives two possible solutions

$$\vec{a}(t) = \begin{cases} 0 \\ \vec{a}(0)e^{t/\tau} \leftarrow \text{unphysical!} \end{cases}$$

clearly,  $\vec{a}(t) = 0$  is the physical solution - a charge at rest, with no force acting on it, should remain at rest - it should not spontaneously accelerate with ~~more~~ exponentially increasing acceleration!

For example, if  $\vec{F}_{ext} = 0$ ,  $\vec{v}(0) = 0$ ,  $\vec{u}(0) = 0$ ,  
 solution of Abraham Lorentz eqn is

$$\vec{a}(t) = \vec{a}(0) e^{t/\tau} \quad \text{"runaway acceleration"}$$

Clearly this solution is unphysical (a charge at rest, with no force acting on it should remain at rest - not spontaneously accelerate!) unless we also explicitly specify  $\vec{a}(0) = 0$ .  
 But even if we reformulate Abraham-Lorentz to remove such runaway solutions, there remain problems.

(see problem 9.30  
 in text, or Jackson)

Reformulate as follows:

$$\begin{aligned} \text{Define } \vec{u}(t) &\equiv \vec{a}(t) e^{-t/\tau} \\ \vec{a}(t) &= \vec{u}(t) e^{t/\tau} \\ \dot{\vec{a}}(t) &= \dot{\vec{u}}(t) e^{t/\tau} + \frac{\vec{u}(t)}{\tau} e^{t/\tau} \end{aligned}$$

$$\Rightarrow \vec{a}(t) - \tau \dot{\vec{a}}(t) = -\tau \dot{\vec{u}} e^{t/\tau}$$

$$\text{Abraham Lorentz} \Rightarrow m(\vec{a} - \tau \dot{\vec{a}}) = -\tau m \dot{\vec{u}} e^{t/\tau} = \vec{F}_{ext}$$

$$\dot{\vec{u}} = -\frac{1}{m\tau} e^{-t/\tau} \vec{F}_{ext}$$

integrate

$$\vec{u}(t) - \vec{u}(t_0) = -\frac{1}{m\tau} \int_{t_0}^t e^{-t'/\tau} \vec{F}_{ext}(t') dt'$$

$$\boxed{m(\vec{a}(t) - \vec{a}(t_0)) = \frac{e^{-t/\tau}}{\tau} \int_t^{t_0} e^{-t'/\tau} \vec{F}_{ext}(t') dt'}$$

~~m(\vec{a}(t))~~

$$\Rightarrow \vec{a}(t) e^{-t/\tau} - \vec{a}(t_0) e^{-t_0/\tau} = \frac{1}{m\tau} \int_{t_0}^t e^{-t'/\tau} \vec{F}_{\text{ext}}(t') dt'$$

if  $\vec{F}_{\text{ext}} = 0$  then  $\vec{a}(t) = \vec{a}(t_0) e^{-(t-t_0)/\tau}$

we will get the desired physical solution  $\vec{a}(t) = 0$   
provided we take  $t_0 \rightarrow \infty$

$$\vec{u}(t) - \vec{u}(\infty) = -\frac{1}{m\tau} \int_{\infty}^t e^{-t'/\tau} \vec{F}_{\text{ext}}(t') dt'$$

$$u(\infty) = \vec{a}(\infty) e^{-\infty/\tau} = 0$$

$$\vec{u}(t) = -\frac{1}{m\tau} \int_{\infty}^t e^{-t'/\tau} \vec{F}_{\text{ext}}(t') dt'$$

$$m \vec{a}(t) = +\frac{1}{\tau} \int_t^{\infty} e^{-\frac{t-t'}{\tau}} \vec{F}_{\text{ext}} dt'$$

change integration variables  $s = \frac{t-t'}{\tau}$   $\tau ds = dt'$

$$m \vec{a}(t) = \int_0^{\infty} ds e^{-s} \vec{F}_{\text{ext}}(t + \tau s)$$

in limit  $\tau \rightarrow 0$   $F_{\text{ext}}(t + \tau s) \approx \vec{F}_{\text{ext}}(t) + \dot{\vec{F}}_{\text{ext}} \tau s + \dots$

$$m \vec{a}(t) = \int_0^{\infty} ds e^{-s} \left[ \vec{F}_{\text{ext}}(t) + \dot{\vec{F}}_{\text{ext}} \tau s + \dots \right]$$

$$= \vec{F}_{\text{ext}}(t) + \tau \dot{\vec{F}}_{\text{ext}} \int_0^{\infty} ds e^{-s} s + o(\tau^2)$$

$\hookrightarrow 0$  as  $\tau \rightarrow 0$

So recover Newton's eqn in limit  $\tau \rightarrow 0$

( $\tau \rightarrow 0 \Rightarrow q \rightarrow 0$  i.e. particle is uncharged  $\therefore$  no radiation-reaction)

Integral-differential form of Abraham-Lorentz eqn

$$m \vec{a}(t) = \int_0^{\infty} ds e^{-s/\tau} \vec{F}_{\text{ext}}(t+\tau s) \quad \left\{ \begin{array}{l} \leftarrow \text{removes all} \\ \text{runaway solutions} \end{array} \right.$$

gives unique solution with only  $\vec{r}(0)$  and  $\vec{v}(0)$  specified

But notice that  $\vec{a}(t)$  depends not on instantaneous force at time  $t$  (as in Newton's eqn), but on the forces ~~that~~ that will act over time  $\tau$  into the future! Above eqn violates causality!

Suppose  $\vec{F}_{\text{ext}} = 0, t < 0$   
 $= \vec{F}_0, t \geq 0$

$$m \vec{a}(t) = \int_0^{\infty} ds e^{-s/\tau} \vec{F}_{\text{ext}}(t+\tau s)$$
$$= \int_{s_{\min}}^{\infty} ds e^{-s/\tau} \vec{F}_0 \quad \text{where } \left\{ \begin{array}{l} s_{\min} = 0 \text{ for } t \geq 0, \text{ or} \\ t + \tau s_{\min} = 0 \Rightarrow \\ s_{\min} = -t/\tau \text{ for } t < 0 \end{array} \right.$$
$$= e^{-s_{\min}/\tau} \vec{F}_0$$

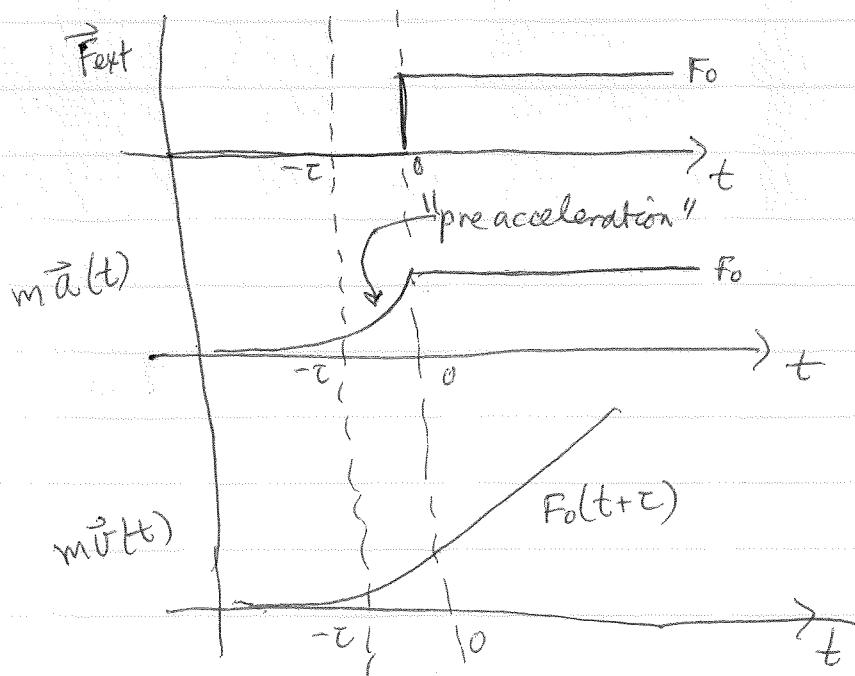
$$m \vec{a}(t) = \left. \begin{array}{l} \vec{F}_0 e^{t/\tau} \quad t < 0 \\ \vec{F}_0 \quad t \geq 0 \end{array} \right\} = m \vec{v}^{\circ}$$

$$\Rightarrow m\vec{v}(t) = \int_{-\infty}^t dt' m\vec{\dot{v}}(t') = \int_{-\infty}^t dt' \vec{F}_0 e^{-s_{\text{ret}}(t-t')}$$

$$\text{for } t < 0 = \int_{-\infty}^t dt' \vec{F}_0 e^{t'/\tau} = \vec{F}_0 \tau e^{t/\tau}$$

$$\text{for } t > 0 = \int_{-\infty}^0 dt' \vec{F}_0 e^{t'/\tau} + \int_0^t dt' \vec{F}_0 = \vec{F}_0 \tau + \vec{F}_0 t = \vec{F}_0 (t + \tau)$$

$$m\vec{v}(t) = \begin{cases} \vec{F}_0 \tau e^{t/\tau} & t < 0 \\ \vec{F}_0 (t + \tau) & t > 0 \end{cases}$$



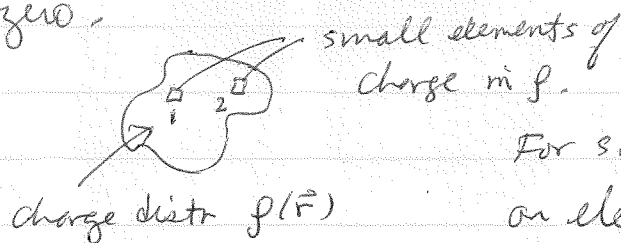
acceleration and velocity start to increase before force is turned on! charge starts to move a time  $\tau$  before the force acts on it.

For classical problems, this "preacceleration" is unimportant as it only occurs for the extremely small time  $\tau$ . However this violation of causality is a conceptual theoretical problem with classical E+M + classical mechanics, that has never been resolved. Quantum mechanics also doesn't resolve this problem!

## Origin of the Radiation-Reaction force

An exact argument which leads to Abraham-Lorentz equation, & explains physical origin of radiation-reaction force is as follows:

Radiation-reaction is the force of a charges  $\vec{E} + \vec{B}$  fields, acting back upon the charge itself. These forces are unimportant in statics, as they always balance out to zero.



For static  $\rho$ , force of element 1 on element 2, is always (-) force of element 2 on element 1.

$\Rightarrow$  Net force acting on  $\rho$ , due to  $\vec{E}$  fields produced by  $\rho$ , is always zero!

This is no longer true in dynamics.

The self force on a charge and current distribution is

Lorentz Force  $\Rightarrow \vec{F}_{\text{self}} = \int d^3r \left\{ \rho(\vec{r}, t) \vec{E}_s(\vec{r}, t) + \vec{j}(\vec{r}, t) \times \vec{B}_s(\vec{r}, t) \right\}$

where  $\vec{E}_s$  and  $\vec{B}_s$  are the "self" electric and magnetic fields produced by  $\rho$  and  $\vec{j}$ .



- 1) Assume  $\rho(\vec{r})$  is spherically symmetric
- 2) Assume  $\rho$  is rigid - doesn't distort as it moves }  $\Rightarrow$  non relativistic approx
- 3) Assume non relativistic motion  $\Rightarrow \vec{j}(\vec{r}, t) = \vec{v}(t) \rho(\vec{r}, t)$
- 4) Assume charge is instantaneously at rest at time  $t_0$  i.e.  $\vec{v}(t_0) = 0$   
(But  $\vec{a}(t_0) \neq 0$  in general) (could always transform to a frame of reference in which this is so)

at time  $t_0$ :

$$\Rightarrow \vec{F}_{\text{self}} = \int d^3r \rho(\vec{r}, t_0) \vec{E}_s(\vec{r}, t_0) \quad \vec{j}(\vec{r}, t_0) = 0 \text{ as } \vec{v}(t_0) = 0$$

substitute in above to compute  $\vec{F}_{\text{self}}$

$$\left. \begin{aligned} \vec{E}_s(\vec{r}, t_0) &= -\vec{\nabla} V_s(\vec{r}, t_0) - \frac{\partial \vec{A}(\vec{r}, t_0)}{\partial t} \\ V_s(\vec{r}, t_0) &= \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\vec{r}', t')}{|\vec{r} - \vec{r}'|} \\ \vec{A}_s(\vec{r}, t_0) &= \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{j}(\vec{r}', t')}{|\vec{r} - \vec{r}'|} \end{aligned} \right\} \begin{aligned} t' &= t_0 - \frac{|\vec{r} - \vec{r}'|}{c} \\ \vec{j}(t') &\neq 0 \text{ in general} \end{aligned}$$

~~Assume particle is small size  $a_0$ , moving slowly so that time it takes for fields to travel across length of particle  $\frac{a_0}{c}$  is much less than the <sup>time</sup> ~~distance~~ the particle takes to move a distance  $d \approx a_0$ , i.e.  $v \ll c$ .  
Then can expand function of  $t'$  about  $t_0$ .~~

$$\vec{F}_{\text{self}}(t_0) = \int d^3r \int d^3r' \rho(\vec{r}, t_0) \left\{ \begin{aligned} &-\vec{\nabla} \left( \frac{\rho(\vec{r}', t')}{|\vec{r} - \vec{r}'|} \right) \\ &- \frac{\mu_0}{4\pi} \frac{\partial}{\partial t} \left( \frac{\vec{j}(\vec{r}', t')}{|\vec{r} - \vec{r}'|} \right) \end{aligned} \right\}$$

$$\vec{F}_{\text{self}} = \int d^3r \int d^3r' \frac{\rho(\vec{r}, t_0)}{4\pi\epsilon_0} \left\{ -\vec{\nabla} \left( \frac{\rho(\vec{r}', t')}{|\vec{r}-\vec{r}'|} \right) - \frac{1}{c^2} \frac{\partial}{\partial t} \left( \frac{\dot{\vec{v}}(t') \rho(\vec{r}', t')}{|\vec{r}-\vec{r}'|} \right) \right\}$$

If charge is moving non-relativistically  $v \ll c$ ,  
 After many expansions + much algebra (see Jackson)  
 one finds

$$\vec{F}_{\text{self}} = \underbrace{-\frac{4}{3} \frac{U}{c^2} \dot{\vec{v}}(t_0)}_{\text{inertial term}} + \underbrace{\frac{1}{4\pi\epsilon_0} \frac{2q^2}{3c^3} \ddot{\vec{v}}(t_0)}_{\text{Abraham Lorentz term}} + (\dots) a_0$$

↑ nonlinear terms in time derivatives of  $\vec{v}$

where  $U \equiv \frac{1}{4\pi\epsilon_0} \frac{1}{2} \int d^3r \int d^3r' \frac{\rho(\vec{r}) \rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$

is electrostatic self energy of the charge

$a_0$  is radius of charge

For a point charge,  $a_0 \rightarrow 0$ , and all terms but first two vanish. Equ of motion becomes

$$m \vec{a} = \vec{F}_{\text{self}} + \vec{F}_{\text{ext}} = -\frac{4}{3} \frac{U}{c^2} \vec{a} + \frac{1}{4\pi\epsilon_0} \frac{2q^2}{3c^3} \dot{\vec{a}} + \vec{F}_{\text{ext}}$$

$$\text{or } \left( m + \frac{4}{3} \frac{U}{c^2} \right) \vec{a} = \frac{1}{4\pi\epsilon_0} \frac{2q^2}{3c^3} \dot{\vec{a}} + \vec{F}_{\text{ext}}$$

two effects: (1) get radiation reaction force  $\vec{F}_{\text{rad}} = m \dot{\vec{a}}$   
 as we argued earlier

(2) "renormalize" inertial mass of the charge

$$m_g = m + \frac{4}{3} \frac{U}{c^2} = m + m_{\text{em}}$$

Define  $\tau = \frac{1}{4\pi\epsilon_0} \frac{2q^2}{3c^3 m_f}$

$$\Rightarrow m_f \vec{a} = m_f \tau \dot{\vec{a}} + \vec{F}_{ext}$$

$$\Rightarrow m_f (\vec{a} - \tau \dot{\vec{a}}) = \vec{F}_{ext}$$

Abraham-Lorentz Equation - only experimental mass  $m_f$  appears - no way to detect  $m$  or  $m_{em}$  separately

$m_f$  = "observed" or "experimental" mass of charge  $q$

$m_{em} = \frac{4}{3} \frac{U}{c^2}$  is the inertial mass associated

with moving the electric and magnetic fields that surround the charge (see HW)

$m_{em}$  is the electromagnetic mass of the ~~part~~ charge

problems with  $m_{em}$

(i)  $m_{em} = \frac{4}{3} \frac{U}{c^2}$  but from relativity, would expect to find  $m_{em} = \frac{U}{c^2}$ . Why the  $\frac{4}{3}$ ?

When assumed  $\rho(\vec{r}, t)$  was rigid in calculation of  $\vec{F}_{self}$ , we neglected the mechanical forces needed to keep this rigidity, i.e. the mechanical forces needed to prevent the small elements of charge that make up the particle from repelling away from each other. These mechanical forces are known as the "Poincaré stresses", and if one includes them, one can make  $m_{em} = U/c^2$  consistent with relativity (read in Jackson Apt 17)

(2) as  $a_0 \rightarrow 0$  (needed to get rid of the high order terms in eqn of motion), we find  $U \rightarrow \infty$  so  $m_{em} \rightarrow \infty$

$$U \sim \frac{1}{4\pi\epsilon_0} \frac{q^2}{a_0}$$

But experimentally observed mass of the charge is  $(m + m_{em})$  is finite!

$\Rightarrow$  mechanical mass  $m \rightarrow -\infty$ ?

If we assumed  $a_0$  was finite, to keep  $U$ , finite,  
and we assumed that all the mass of the ~~electron~~ charge  
was electromagnetic in origin (ie  $m=0$ ), then

$$a_0 \sim \frac{q^2}{4\pi\epsilon_0 mc^2} = 2.8 \times 10^{-13} \text{ cm for the electron}$$

"classical electron radius"

But it is well established that electron is a point  
particle at least down to lengths  $10^{-15}$  cm!

Note, classical electron radius

$$a_0 \sim \tau c \quad \text{where } \tau = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2}{mc^3}$$

length scale on which radiation-reaction effects  
are no longer a small perturbation.

Conclusion: Classical mechanics + classical electro-  
magnetism work just fine, provided one  
stays in the regime where radiation-  
reaction effects remain only a small  
perturbation on the motion of charges.  
If one tries to extend the theory down  
to length and time scales smaller than  
this (ie  $t < \tau$  or  $l < c\tau$ ) fundamental  
problems arise having to do with ~~what~~ the  
~~is meant by~~ <sup>definition of</sup> a "point" charge. These  
problems have never been satisfactorily  
resolved (Nobel prize is awaiting!)