Statics Review

charge density $\rho(\vec{r})$

$$\int_{\text{volume } V} d^3r\, \rho(\vec{r}) = Q \quad \text{total charge inside volume } V$$

current density $\vec{J}(\vec{r})$

$$\int_{\text{surface } S} d\vec{a}\cdot\vec{J}(\vec{r}) = I \quad \text{total current (charge per unit time) flowing through surface } S$$

local charge conservation

$$\frac{\partial \rho(\vec{r})}{\partial t} + \vec{\nabla}\cdot\vec{J}(\vec{r}) = 0$$

in electrostatics, $\frac{\partial \rho}{\partial t} = 0 \Rightarrow \vec{\nabla}\cdot\vec{J}(\vec{r}) = 0$ is key condition for magnetostatics

– also $\frac{\partial \vec{J}}{\partial t} = 0$

Maxwell's Equations

electrostatics | **magnetostatics**

integral form:

Gauss $\oint_{\text{surface } S} \vec{E}\cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_0}$ $\qquad$ $\oint_S \vec{B}\cdot d\vec{a} = 0$

$\oint_{\text{curve } C} \vec{E}\cdot d\vec{l} = 0$ $\qquad$ Ampere $\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}$

differential form:

$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_0}$ $\qquad$ $\vec{\nabla}\cdot\vec{B}(\vec{r}) = 0$

$\vec{\nabla}\times\vec{E}(\vec{r}) = 0$ $\qquad$ $\vec{\nabla}\times\vec{B}(\vec{r}) = \mu_0\vec{J}(\vec{r})$

Potentials

electrostatic potential $V(\vec{r})$

$$\vec{E} = -\vec{\nabla} V$$

magnetostatic vector potential $\vec{A}(\vec{r})$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

can also choose $\vec{A}$ to satisfy the additional condition $\vec{\nabla} \cdot \vec{A} = 0$ see Griffiths sec 5.4.1

"Coulomb Gauge"

If one has an $\vec{A}$ such that $\vec{B} = \vec{\nabla} \times \vec{A}$ but $\vec{\nabla} \cdot \vec{A} \neq 0$, we can always transform to a new $\vec{A}' = \vec{A} + \vec{\nabla}\lambda$ where $\lambda(\vec{r})$ is any scalar function. Then

$$\vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} + \underbrace{\vec{\nabla} \times \vec{\nabla}\lambda}_{=0 \text{ always}} = \vec{\nabla} \times \vec{A} = \vec{B}$$

Can we choose λ so that $\vec{\nabla} \cdot \vec{A}' = 0$? yes!

$$\vec{\nabla} \cdot \vec{A}' = \vec{\nabla} \cdot \vec{A} + \nabla^2 \lambda = 0 \Rightarrow \nabla^2 \lambda = -\vec{\nabla} \cdot \vec{A}$$

has same form as $\nabla^2 V = -\rho/\epsilon_0$ so there is a solution for λ

$$\lambda(\vec{r}) = \frac{1}{4\pi} \int d^3r' \, \frac{\vec{\nabla} \cdot \vec{A}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

So we can always find a vector potential so that

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \underline{\text{AND}} \quad \vec{\nabla} \cdot \vec{A} = 0.$$

Maxwell's equations for statics in potential form

$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ and $\vec{E} = -\vec{\nabla}V \Rightarrow \nabla^2 V = -\rho/\varepsilon_0$

$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ and $\vec{B} = \vec{\nabla}\times\vec{A}$

$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J}$

in Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$ so $\nabla^2\vec{A} = -\mu_0\vec{J}$

solutions for localized sources

$$\nabla^2 V = -\rho/\varepsilon_0 \Rightarrow V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \, \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

$$\nabla^2\vec{A} = \mu_0\vec{J} \Rightarrow \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

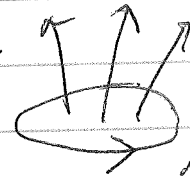
Magnetic Induction and Faraday's Law

A changing magnetic flux through a loop induces an electromotive force (emf) around the loop

$$\mathcal{E} = -\frac{d\Phi}{dt} \qquad \Phi = \int \vec{B}\cdot d\vec{a}$$ flux of $\vec{B}$ through the loop

$$\mathcal{E} = \oint \vec{E}\cdot d\vec{l}$$ circulation of $\vec{E}$ around the loop

Must take directions consistent with the right hand rule.



← if go around loop as shown then flux must be computed in direction as shown

$$\mathcal{E} = -\frac{d\Phi}{dt} \Rightarrow \oint_C \vec{E}\cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}$$

S is any surface bounded by the curve C

By Stokes theorem

$$\oint_C \vec{E}\cdot d\vec{l} = \int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{a}$$

So $$\int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{a} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}$$

Must be true for ANY surface S (does not need to be a physical loop at C)

Faraday's Law

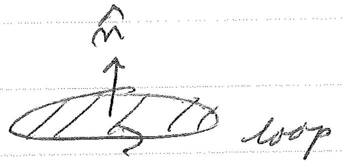
$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

When $\frac{\partial \vec{B}}{\partial t} = 0$ then recover $\vec{\nabla}\times\vec{E} = 0$

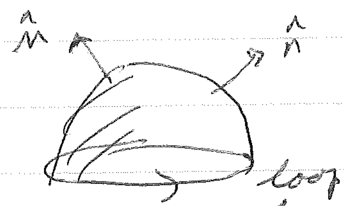
Magnetic Flux

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

When we compute the flux through a loop, does it matter what surface we use to compute Φ?



Surface S is flat surface in plane of loop



Surface S′ is hemisphere bounded by loop.

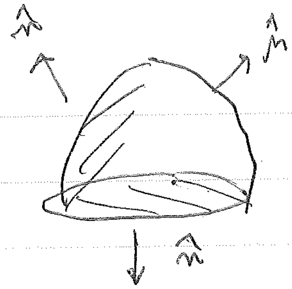
$$\int_S d\vec{a} \cdot \vec{B} \overset{?}{=} \int_{S'} d\vec{a} \cdot \vec{B}$$

YES!! all surfaces bounded by the same loop will give the same $\Phi = \int d\vec{a} \cdot \vec{B}$

Follows from fact that $\vec{\nabla} \cdot \vec{B} = 0$. To see this:

method I: Consider the two surfaces S and S′ as above. Construct the closed surface $S'' = S' - S$ where $-S$ is the same as S but with $\hat{n}$ in the opposite direction.

S'' is just the hemisphere S' closed off by the planar S at the bottom



Gauss' Theorem

$$\oint_{S''} d\vec{a}\cdot\vec{B} \overset{\downarrow}{=} \int_{V} d^3r\ \vec{\nabla}\cdot\vec{B} = 0 \quad \text{since } \vec{\nabla}\cdot\vec{B}=0$$

$$\oint_{S''} d\vec{a}\cdot\vec{B} = \oint_{S'} d\vec{a}\cdot\vec{B} - \oint_{S} d\vec{a}\cdot\vec{B} = 0$$

$$\Rightarrow \oint_{S} d\vec{a}\cdot\vec{B} = \oint_{S''} d\vec{a}\cdot\vec{B}$$ same through both surfaces

method II

Since $\vec{\nabla}\cdot\vec{B}=0$, we can write $\vec{B}=\vec{\nabla}\times\vec{A}$

Then $$\Phi = \int_{S} d\vec{a}\cdot\vec{B} = \int_{S} d\vec{a}\cdot(\vec{\nabla}\times\vec{A}) = \oint_{C} d\vec{l}\cdot\vec{A}$$

by Stokes theorem, where C is the curve bounding S.

Now $\oint_C d\vec{l}\cdot\vec{A}$ depends only on the curve C, not on the surface S, so Φ does not depend on S - only C.

Does Φ depend on what gauge we used for $\vec{A}$? No!

$\vec{A}' = \vec{A} + \vec{\nabla}\lambda$, λ any scalar function. Then

$$\oint_C d\vec{l}\cdot\vec{A}' = \oint_C d\vec{l}\cdot\vec{A} + \underbrace{\oint_C d\vec{l}\cdot\vec{\nabla}\lambda}_{=0}$$

So $\oint_C d\vec{l}\cdot\vec{A}$ is the same for all $\vec{A}$ that give the same $\vec{B}$

Energy stored in a static configuration of charges

point charges q_i at positions $\vec{r}_i$. Imagine starting with all q_i separated at infinity and compute the work done to bring them one by one to their positions at $\vec{r}_i$.

$$W = \underbrace{0}_{\text{move in } q_1} + \underbrace{\frac{q_1 q_2}{4\pi\varepsilon_0 |\vec{r}_1 - \vec{r}_2|}}_{\text{move in } q_2} + \underbrace{\frac{q_1 q_3}{4\pi\varepsilon_0 |\vec{r}_1 - \vec{r}_3|} + \frac{q_2 q_3}{4\pi\varepsilon_0 |\vec{r}_2 - \vec{r}_3|}}_{\text{move in } q_3}$$

$$+ \underbrace{\frac{q_1 q_4}{4\pi\varepsilon_0 |\vec{r}_1 - \vec{r}_4|} + \frac{q_2 q_4}{4\pi\varepsilon_0 |\vec{r}_2 - \vec{r}_4|} + \frac{q_3 q_4}{4\pi\varepsilon_0 |\vec{r}_3 - \vec{r}_4|}}_{\text{move in } q_4} + \cdots$$

$$= \sum_{i<j} \frac{q_i q_j}{4\pi\varepsilon_0 |\vec{r}_i - \vec{r}_j|} = \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{4\pi\varepsilon_0 |\vec{r}_i - \vec{r}_j|}$$

generalize to continuous charge density $\rho(\vec{r})$

$$W = \frac{1}{2} \int d^3r \int d^3r' \, \frac{\rho(\vec{r})\rho(\vec{r}')}{4\pi\varepsilon_0 |\vec{r} - \vec{r}'|}$$

$$= \frac{1}{2} \int d^3r \, \rho(\vec{r}) V(\vec{r}) \qquad \text{where } V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \, \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

is electrostatic potential

$$\nabla^2 V = -\rho/\varepsilon_0$$

Now use $\rho = \varepsilon_0 \vec{\nabla} \cdot \vec{E}$

$$W = \frac{\varepsilon_0}{2} \int d^3r \, (\vec{\nabla} \cdot \vec{E}) V \qquad \text{use } \vec{\nabla} \cdot (\vec{E} V) = (\vec{\nabla} \cdot \vec{E}) V + \vec{E} \cdot \vec{\nabla} V$$

$$= \frac{\varepsilon_0}{2} \int d^3r \, \left(\vec{\nabla} \cdot (\vec{E} V) - \vec{E} \cdot \vec{\nabla} V \right)$$

use $\vec{E} = -\vec{\nabla} V$ and $\int d^3r\, \vec{\nabla}\cdot(\vec{E}V) = \oint_S d\vec{a}\cdot\vec{E}V$

by Gauss' Theorem

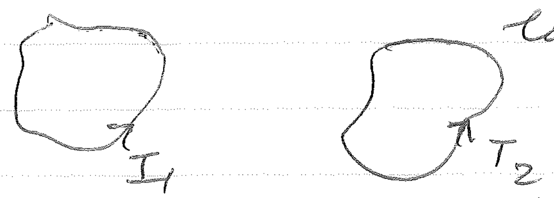
$$W = \frac{\varepsilon_0}{2}\oint_S d\vec{a}\cdot\vec{E}V + \frac{\varepsilon_0}{2}\int d^3r\, |\vec{E}|^2$$

as volume of integration fills all space, surface $S \to \infty$ where $\vec{E}, V \to 0$ (assumes charge distribution $\rho(\vec{r})$ is localized) $\Rightarrow$ 1st term vanishes

$$W = \frac{\varepsilon_0}{2}\int d^3r\, |\vec{E}|^2$$

magnetostatic

<u>Energy stored in a configuration of current carrying loops</u>



What is the work done to create this configuration of current carrying loops? This will be the energy stored in the configuration. We would like this energy to be <u>independent</u> of the process used to construct the configuration.

loops initially at infinity with $I_1 = I_2 = 0$.

<u>Method I</u> 1) Keeping $I_1 = I_2 = 0$, move the loops in from infinity to their final positions.

Since $I_1 = I_2 = 0$, the loops exert no forces on each other, so the work done in this step is zero.

2) Now, with loops in their final positions, turn the currents up from zero to I_1 and I_2

Step (2) costs work as follows: As I_1 and I_2 change in time, they give rise to changing magnetic flux through the loops, which gives rise to back emfs around the