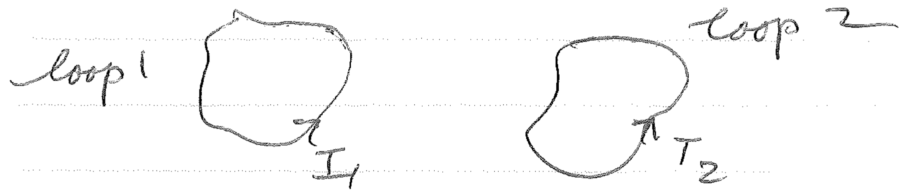
magnetostatic

Energy stored in a configuration of current carrying loops



What is the work done to create this configuration of current carrying loops? This will be the energy stored in the configuration. We would like this energy to be <u>independent</u> of the process used to construct the configuration.

loops initially at infinity with $I_1 = I_2 = 0$.

Method I

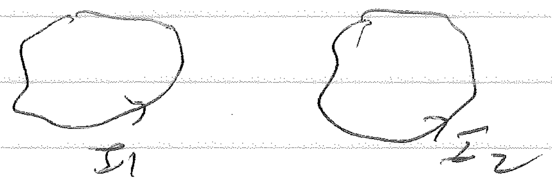
1) Keeping $I_1 = I_2 = 0$, move the loops in from infinity to their final positions.

Since $I_1 = I_2 = 0$, the loops exert no forces on each other, so the work done in this step is zero.

2) Now, with loops in their final positions, turn the currents up from zero to I_1 and I_2

Step (2) costs work as follows: As I_1 and I_2 change in time, they give rise to changing magnetic flux through the loops, which gives rise to back emfs around the

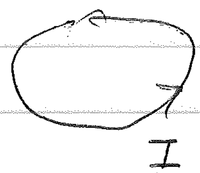
Mutual and self inductance



current in loop 1 gives magnetic flux through loop 2

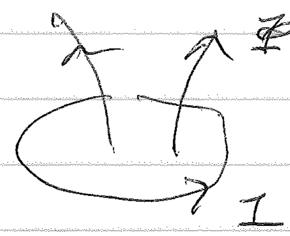
$\Phi_2 = M_{21} I_1$ $\qquad M_{21} = \frac{\mu_0}{4\pi} \oint_1 \oint_2 \frac{d\vec{l}_1 \cdot d\vec{l}_2}{|\vec{r}_1 - \vec{r}_2|}$

$\Phi_1 = M_{12} I_2$ $\qquad M_{12} = M_{21}$



$\Phi = L I$ $\qquad L = \frac{\mu_0}{4\pi} \oint_1 \oint_1 \frac{d\vec{l} \cdot d\vec{l}'}{|\vec{r} - \vec{r}'|}$

Right hand rule



Changing I_1 gives changing Φ_2

$$\frac{d\Phi_2}{dt} = M_{21} \frac{dI_1}{dt} = -\mathcal{E}_2$$ induced emf in loop 2

loops, which acts to oppose the currents I_1 and I_2.
To keep the currents I_1 and I_2 flowing we have to do work against this induced emf.

emf in loop 1: $\mathcal{E}_1 = \underbrace{-L_1 \frac{dI_1}{dt}}_{\text{self inductance}} \underbrace{-M \frac{dI_2}{dt}}_{\text{mutual inductance}}$

emf in loop 2: $\mathcal{E}_2 = -L_2 \frac{dI_2}{dt} - M \frac{dI_1}{dt}$

To keep I_1 and I_2 increasing, we must due work <u>against</u> these induced back emfs

$$\frac{dW}{dt} = -\mathcal{E}_1 I_1 - \mathcal{E}_2 I_2$$

rate of doing work

work per charge in one trip around loop

charge per time going around loop

$$\frac{dW}{dt} = \left(L_1 \frac{dI_1}{dt}\right) I_1 + \left(M \frac{dI_2}{dt}\right) I_1 + \left(L_2 \frac{dI_2}{dt}\right) I_2 + \left(M \frac{dI_1}{dt}\right) I_2$$

$$\frac{dW}{dt} = \frac{1}{2} L_1 \frac{d}{dt}(I_1^2) + \frac{1}{2} L_2 \frac{d}{dt}(I_2^2) + M \frac{d}{dt}(I_1 I_2)$$

$$W = \int_0^T dt \frac{dW}{dt} = \underbrace{\frac{1}{2} L_1 I_1^2}_{\text{self energy loop 1}} + \underbrace{\frac{1}{2} L_2 I_2^2}_{\text{self energy loop 2}} + \underbrace{M I_1 I_2}_{\text{interaction energy}}$$

Substitute in our expressions for L_1, L_2, M to get

$$W = \frac{1}{2}\frac{\mu_0}{4\pi} I_1^2 \oint_1 \oint_1 \frac{d\vec{l}\cdot d\vec{l}'}{|\vec{r}-\vec{r}'|} + \frac{1}{2}\frac{\mu_0}{4\pi} I_2^2 \oint_2 \oint_2 \frac{d\vec{l}\cdot d\vec{l}'}{|\vec{r}-\vec{r}'|}$$

$$+ \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{d\vec{l}\cdot d\vec{l}'}{|\vec{r}-\vec{r}'|}$$

use $I\,d\vec{l} = \vec{I}\,dl$ since current in wire is always tangential to wire

$$W = \frac{1}{2}\frac{\mu_0}{4\pi}\left\{ \oint_1 \oint_1 dl\,dl' \frac{\vec{I}(\vec{r})\cdot\vec{I}(\vec{r}')}{|\vec{r}-\vec{r}'|} + \oint_2 \oint_2 dl\,dl' \frac{\vec{I}(\vec{r})\cdot\vec{I}(\vec{r}')}{|\vec{r}-\vec{r}'|} \right.$$

$$\left. + 2 \oint_1 \oint_2 dl\,dl' \frac{\vec{I}(\vec{r})\cdot\vec{I}(\vec{r}')}{|\vec{r}-\vec{r}'|} \right\}$$

where $\vec{I}(\vec{r})$ is current at position $\vec{r}$ on loop.

$$W = \frac{1}{2}\frac{\mu_0}{4\pi} \oint_{1+2} \oint_{1+2} dl\,dl' \frac{\vec{I}(\vec{r})\cdot\vec{I}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

where $\oint_{1+2} = \oint_1 + \oint_2$

We can now see the generalization to n loops with currents $I_1, \ldots, I_n$

$$W = \frac{1}{2}\frac{\mu_0}{4\pi} \oint_{1+2+\dots+n} \oint_{1+2+\dots+n} dl\, dl' \; \frac{\vec{I}(\vec{r})\cdot\vec{I}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

and then the generalization to a continuous current density $\vec{j}(\vec{r})$

$$W = \frac{1}{2}\frac{\mu_0}{4\pi} \int d^3r \int d^3r' \; \frac{\vec{j}(\vec{r})\cdot\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

has a similar form to the energy stored in an electrostatic configuration with charge density $\rho(\vec{r})$

$$W = \frac{1}{2}\frac{1}{4\pi\varepsilon_0} \int d^3r \int d^3r' \; \frac{\rho(\vec{r})\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

For a magnetostatic $\vec{j}(\vec{r})$, we can express the magnetic vector potential, in the Coulomb gauge $\vec{\nabla}\cdot\vec{A}=0$, as

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \; \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

(follows from $-\nabla^2\vec{A} = \mu_0\vec{j}$ in Coulomb gauge)

So then we have

$$W = \frac{1}{2} \int d^3r \; \vec{j}(\vec{r})\cdot\vec{A}(\vec{r})$$

Compare to similar for for electrostatic energy

$$W = \frac{1}{2} \int d^3r \; \rho(\vec{r})\, V(\vec{r})$$

Finally, for a magnetostatic current $\vec{J}$ we have Ampere's Law

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

So we can write

$$W = \frac{1}{2\mu_0} \int d^3r \; \vec{A}(\vec{r}) \cdot \left[\vec{\nabla} \times \vec{B}(\vec{r}) \right]$$

integrate by parts using $\vec{\nabla} \cdot (A \times B) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$

$$W = \frac{1}{2\mu_0} \int d^3r \left[\vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{\nabla} \cdot (\vec{A} \times \vec{B}) \right]$$

Gauss theorem

$$= \frac{1}{2\mu_0} \int d^3r \; |\vec{B}|^2 - \frac{1}{2\mu} \oint_{\text{surface}} d\vec{a} \cdot (\vec{A} \times \vec{B})$$

Now let the surface $\to$ infinity. For localized sources, $\vec{A}, \vec{B}$ will vanish sufficiently fast as $\vec{r} \to \infty$ so that the surface integral will vanish. We are then left with

$$W = \frac{1}{2\mu_0} \int d^3r \; |\vec{B}(\vec{r})|^2$$

Compared to electrostatic energy

$$W = \frac{\varepsilon_0}{2} \int d^3r \; |\vec{E}(\vec{r})|^2$$

Summary

magnetostatic energy	electrostatic energy
$W = \frac{1}{2}\frac{\mu_0}{4\pi}\int d^3r \int d^3r' \frac{\vec{j}(\vec{r})\cdot\vec{j}(\vec{r}')}{\lvert\vec{r}-\vec{r}'\rvert}$	$W = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\int d^3r \int d^3r' \frac{\rho(\vec{r})\rho(\vec{r}')}{\lvert\vec{r}-\vec{r}'\rvert}$
$W = \frac{1}{2}\int d^3r\, \vec{A}(\vec{r})\cdot\vec{j}(\vec{r})$	$W = \frac{1}{2}\int d^3r\, V(\vec{r})\rho(\vec{r})$
$W = \frac{1}{2\mu_0}\int d^3r\, \lvert\vec{B}(\vec{r})\rvert^2$	$W = \frac{1}{2}\varepsilon_0\int d^3r\, \lvert\vec{E}(\vec{r})\rvert^2$

Method ②

Alternative Way to construct the configuration of current carrying loops

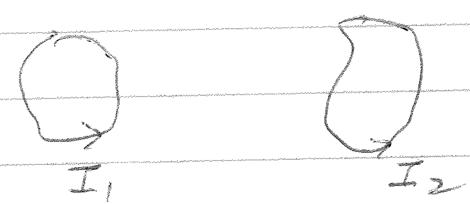
i) Start with loops infinitely far apart with zero current

ii) turn up currents to I_1 and I_2. This costs work

$\frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2$ by previous arguments

iii) move loops in from infinity to final positions keeping I_1 and I_2 held constant.

What is the work done in step (iii)?



Let us move loop 1 into its final position, keeping loop 2 at infinity. This causes no work as the loops stay infinitely apart. Now move loop 2 into its final position. As move loop 2, there is a magneto static force acting on it from the field $\vec{B}_1$ produced by the current in loop 1.

Force on moving charge is $q\vec{v} \times \vec{B}$

Force on element of current density is $d^3r\, \vec{J}(\vec{r}) \times \vec{B}(\vec{r})$

Force on element of current in a wire is $I\, d\vec{l} \times \vec{B}$

Force on loop 2 due to magnetic field $\vec{B}_1$ of loop 1 is

$$\vec{F}_2 = I_2 \oint_2 d\vec{l}_2 \times \vec{B}_1(\vec{r}_2)$$

use Biot-Savart Law $\vec{B}_1(\vec{r}_2) = \frac{\mu_0 I_1}{4\pi} \oint_1 d\vec{l}_1 \times \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}$

so

$$\vec{F}_2 = \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 d\vec{l}_2 \times \frac{(d\vec{l}_1 \times (\vec{r}_2 - \vec{r}_1))}{|\vec{r}_2 - \vec{r}_1|^3}$$

Now use triple product rule $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

$$d\vec{l}_2 \times (d\vec{l}_1 \times (\vec{r}_2 - \vec{r}_1)) = d\vec{l}_1 \, (d\vec{l}_2 \cdot (\vec{r}_2 - \vec{r}_1)) - (\vec{r}_2 - \vec{r}_1)(d\vec{l}_1 \cdot d\vec{l}_2)$$

$$\vec{F}_2 = \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 d\vec{l}_1 \, d\vec{l}_2 \cdot \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}$$

$$- \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 d\vec{l}_1 \cdot d\vec{l}_2 \, \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}$$

first term can be written as

$$-\frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 d\vec{l}_1 \, d\vec{l}_2 \cdot \vec{\nabla}_2 \left(\frac{1}{|\vec{r}_2 - \vec{r}_1|} \right)$$

Now $\oint_2 d\vec{l}_2 \cdot \vec{\nabla}_2 \left(\frac{1}{|\vec{r}_2 - \vec{r}_1|} \right) = 0$ as line integral of any gradient around a closed loop vanishes

So $\vec{F}_2 = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 (d\vec{l}_1 \cdot d\vec{l}_2) \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}$

Now let $\vec{r}_2 = \vec{R} + \delta\vec{r}_2$ where $\vec{R}$ is the center of loop 2. As loop 2 moves, $\vec{R}$ goes from infinity to its final position $\vec{R}_0$

$$\vec{F}_2 = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 (d\vec{l}_1 \cdot d\vec{l}_2) \frac{(\vec{R} + \delta\vec{r}_2 - \vec{r}_1)}{|\vec{R} + \delta\vec{r}_2 - \vec{r}_1|^3}$$

$$= \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 (d\vec{l}_1 \cdot d\vec{l}_2) \vec{\nabla}_R \left(\frac{1}{|\vec{R} + \delta\vec{r}_2 - \vec{r}_1|} \right)$$

derivative with respect to $\vec{R}$,

$\vec{F}_2$ is force acting on loop 2 from loop 1. To move loop 2 into position, the person moving loop 2 must exert a force equal and opposite to $\vec{F}_2$. The work done moving the loop is therefore

$$\tilde{W} = -\int_\infty^{\vec{R}_0} \vec{F}_2 \cdot d\vec{R} \qquad \text{(mover applies force } -\vec{F}_2\text{)}$$

$$= -\frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 (d\vec{l}_1 \cdot d\vec{l}_2) \underbrace{\int_\infty^{\vec{R}_0} d\vec{R} \cdot \vec{\nabla} \left(\frac{1}{|\vec{R} + \delta\vec{r}_2 - \vec{r}_1|} \right)}$$

can easily do the line integral of a gradient

$$= -\frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{(d\vec{l}_1 \cdot d\vec{l}_2)}{|\vec{R}_0 + \delta\vec{r}_2 - \vec{r}_1|}$$

$$\tilde{W} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{d\vec{l}_1 \cdot d\vec{l}_2}{|\vec{r}_2 - \vec{r}_1|} \qquad \vec{r}_2 = \vec{R}_0 + \delta\vec{r}_2$$

integrals are over final positions of loops 1 and 2 compare with our derivation of the mutual inductance M and we see

$$\tilde{W} = -MI_1I_2$$

Thus this method appears to give as the work done to create the current carrying loops in their final position

method ② $$W = \tfrac{1}{2}L_1I_1^2 + \tfrac{1}{2}L_2I_2^2 - MI_1I_2 \quad !!$$

Compare to our first method which gave

method ① $$W = \tfrac{1}{2}L_1I_1^2 + \tfrac{1}{2}L_2I_2^2 + MI_1I_2$$

In method ①, the interaction energy of the loops was MI_1I_2

In method ②, it appears that the interaction energy of the two loops is $-MI_1I_2$

How do we reconcile these two answers?
The energy stored in the configuration of current carrying loops should not depend on the process used to construct the configuration if energy is conserved!

Solution: when we constructed the configuration according to method ② we assumed the currents I_1 and I_2 in the two loops stayed constant as the loop 2 was moved into position with respect to loop 1. But as loop 2 moves, the flux through loop 2 due to current in loop 1 changes $\Rightarrow$ emf induced in loop 2. Similarly, flux through loop 1 changes $\Rightarrow$ emf induced in loop 1.

If we want to keep I_1 and I_2 constant there must be some battery in each loop doing work to counter these induced emfs. The work done by these batteries is

$$\frac{dW_{battery}}{dt} = -\mathcal{E}_1 I_1 - \mathcal{E}_2 I_2 \qquad \mathcal{E}_1 = \frac{-d\Phi_1}{dt}$$

$$= I_1 \frac{d\Phi_1}{dt} + I_2 \frac{d\Phi_2}{dt} \qquad \mathcal{E}_2 = -\frac{d\Phi_2}{dt}$$

$$W_{battery} = \int_0^T dt \left(I_1 \frac{d\Phi_1}{dt} + I_2 \frac{d\Phi_2}{dt} \right) = I_1 \Phi_1 + I_2 \Phi_2$$

integrate from $t=0$ when loops infinitely separated to $t=T$ when loops in final position.

As loop moves, I_1 and I_2 stay constant but Φ_1 and Φ_2 change

$$W_{battery} = I_1 \Phi_1 + I_2 \Phi_2$$

Φ_1 is flux through loop 1 due to field from loop 2.

$$= I_1 (M I_2) + I_2 (M I_1)$$

$$= 2 M I_1 I_2$$

Total work is $\tilde{W} + W_{battery} = -M I_1 I_2 + 2 M I_1 I_2 = M I_1 I_2$

Moral: Even in a magnetostatic configuration such as two loops with steady currents, we need to know about dynamics, ie Faraday's law, in order to compute the magnetostatic energy stored in the configuration