Maxwell's Correction to Ampere's Law

So far: $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$, $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{\nabla}\times\vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

Amperes law for magnetostatics is $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$

But this cannot remain correct more generally since

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0 \vec{\nabla}\cdot\vec{J} = -\mu_0 \frac{\partial \rho}{\partial t}$$

since by charge conservation $\dfrac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$

But $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0$ for any vector function $\vec{B}$

So Amperes law $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ can only be correct when $\dfrac{\partial \rho}{\partial t} = 0$, ie only for statics.

Maxwell argued that Amperes law must have another term in it that vanishes for statics, but makes $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0$ when one is not static. To see what that term is note:

$$\rho = \varepsilon_0 \vec{\nabla}\cdot\vec{E} \Rightarrow \frac{\partial \rho}{\partial t} = \varepsilon_0 \vec{\nabla}\cdot\left(\frac{\partial \vec{E}}{\partial t}\right)$$

$$\text{So } \frac{\partial \rho}{\partial t} - \varepsilon_0 \vec{\nabla}\cdot\left(\frac{\partial \vec{E}}{\partial t}\right) = -\vec{\nabla}\cdot\left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = 0$$

Ampere's law now is: $\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t}\right)$

Now take divergence of both side to consistently get $0 = 0$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \underbrace{\mu_0 \varepsilon \frac{\partial \vec{E}}{\partial t}}_{\text{Maxwells correction}}$$

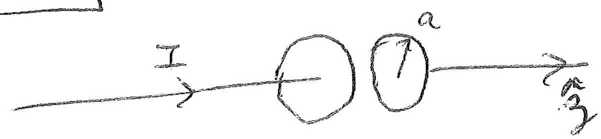
$\varepsilon_0 \dfrac{\partial \vec{E}}{\partial t}$ = "displacement current" or in integral form

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{encl} + \mu_0 \varepsilon_0 \int_S \frac{\partial \vec{E}}{\partial t}\cdot d\vec{a}$$

← surface bounded by loop C

$\frac{\partial \vec{E}}{\partial t}$ is a source of $\vec{B}$ just like $\frac{\partial \vec{B}}{\partial t}$ is a source of $\vec{E}$

7.35 ;



constant I.

$$\nabla \times B = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}$$

$$\oint d\vec{l} \cdot \vec{B} = \mu_0 I_{encl} + \mu_0 \varepsilon_0 \int d\vec{a} \cdot \frac{\partial \vec{E}}{\partial t}$$

find B between plates for $r < a$.

Charge on plates $Q = It$ on left plate, $-Q = -It$ on right plate

$\Rightarrow$ E between plates is $\vec{E} = \frac{\sigma}{\varepsilon_0} \hat{z} = \frac{Q}{\pi a^2 \varepsilon_0} \hat{z}$

parallel plate capacitor

$$\vec{E} = \frac{It}{\varepsilon_0 \pi a^2} \hat{z} \rightarrow \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{I}{\pi a^2} \hat{z}$$

like a current density flowing in $\hat{z}$ direction.

in region between plates, symmetry $\Rightarrow \vec{B} = B(r)\hat{\phi}$

Take loop of radius r centered about wire, inbetween plates

$$\oint d\vec{l} \cdot \vec{B} = 2\pi r B(r) = \mu_0 \underset{\substack{\parallel \\ 0}}{I_{encl}} + \mu_0 \varepsilon_0 \int d\vec{a} \cdot \frac{I}{\varepsilon_0 \pi a^2} \hat{z}$$

area of loop $= \pi r^2$

$$= \mu_0 \varepsilon_0 \frac{\pi r^2 I}{\varepsilon_0 \pi a^2}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0 r}{2\pi a^2} I \hat{\phi}$$

just like we had for a wire with uniform current density

$$\vec{J} = \frac{I}{\pi a^2} \hat{z}$$

for $r > a$, if ignore "edge" effects from non uniformity of $\vec{E}$ at edges of plates

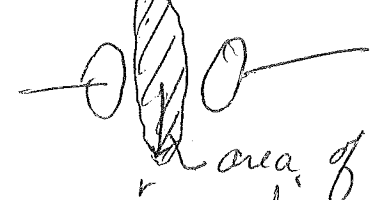
$$\oint d\vec{\ell} \cdot \vec{B} = 2\pi r\, B(r) = \mu_0 \underset{\overset{\shortparallel}{0}}{I_{encl}} + \cancel{\mu_0\varepsilon_0 \int d\vec{a} \cdot \frac{I}{\varepsilon_0 \pi a^2}\hat{x}}$$

$$+ \mu_0 \varepsilon_0 (\pi a^2) \frac{I}{\varepsilon_0 \pi a^2}$$

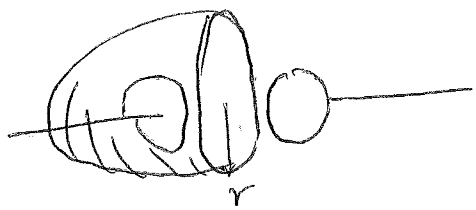
since only area between plates has $\frac{\partial E}{\partial t} \neq 0$

$$= \mu_0 I$$

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{2\pi r}\hat{\phi}$$ just like around wire with current I

to do above, took area of loop as 

but could also take any area bounded by curve,



now $\int d\vec{a} \cdot \frac{\partial \vec{E}}{\partial t} = 0$ on this area

but $I_{encl} = I$

So $B = \frac{\mu_0 I \hat{\phi}}{2\pi r}$ as before

we need Maxwell's displacement current to get consistent results! independent of what surface is used for the integration

Energy and Momentum Conservation

in electrostatics $\quad W_{elec} = \frac{\varepsilon_0}{2}\int d^3r\, E^2 = \frac{1}{2}\int d^3r\, \rho V$ $\quad$ statics only!

in magnetostatics $\quad W_{mag} = \frac{1}{2\mu_0}\int d^3r\, B^2 = \frac{1}{2}\int d^3r\, \vec{j}\cdot\vec{A}$

Now we treat the full electrodynamic situation

Maxwell's Equations

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \nabla\times\vec{B} = \mu_0\vec{j} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

Consider the power dissipated by an electric current flowing in a wire

power $= VI$ $\qquad$ V is voltage drop $= \mathcal{E}$ emf

I is current, charge/time

$V = EL$

E in wire, L length of wire

$I = Aj$ $\quad$ j current density, A cross sectional area

$$\text{power} = VI = (EL)(Aj) = (Ej)(LA)$$

$$= (Ej)(\text{volume})$$

If energy is conserved, then this energy dissipated by the flowing current must go somewhere.

Where does it go? Goes into heating up the wire "Joule heating". What does that heat correspond to mechanically? Increased kinetic energy of the particles in the wire!

Let W_{mech} be the total mechanical energy of particles in some volume of space

$$W_{mech} = (\text{kinetic energy} + \text{potential energy})$$

then expect $$\boxed{\frac{dW_{mech}}{dt} = \int_{vol} d^3r\, \vec{E}\cdot\vec{J}}$$

Another way: Let W_{mech} is the mechanical energy of a collection of charged particles. Then the work-energy theorem of mechanics says the change in energy of a charge q is given by the forces acting on q,

$$dW = \vec{F}\cdot d\vec{r} = (q\vec{E} + q\vec{v}\times\vec{B})\cdot d\vec{r}$$

$$\frac{dW}{dt} = \vec{F}\cdot\frac{d\vec{r}}{dt} = (q\vec{E} + q\vec{v}\times\vec{B})\cdot\vec{v}$$

$$= q\vec{v}\cdot\vec{E}$$

Now add up over all charges in the volume

$$\frac{dW_{mech}}{dt} = \sum_{q_i \text{ in vol}} q_i \vec{v}_i \cdot E(\vec{r}_i)$$

$$= \int_{vol} d^3r \; \vec{J}(\vec{r}) \cdot \vec{E}(\vec{r})$$

where we used $\vec{J}(\vec{r}) = \sum_i q_i \vec{v}_i \, \delta(\vec{r} - \vec{r}_i)$

$$\boxed{\frac{dW_{mech}}{dt} = \int_{vol} d^3r \; \vec{J} \cdot \vec{E}}$$

$$\frac{dW_m}{dt} = \int d^3r\, \vec{J}\cdot\vec{E}$$

Ampere's Law $\vec{J} = \frac{\vec{\nabla}\times\vec{B}}{\mu_0} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

$$\frac{dW_m}{dt} = \int d^3r \left[\underbrace{\frac{\vec{E}\cdot(\vec{\nabla}\times\vec{B})}{\mu_0}}_{\substack{\text{integrate} \\ \text{by parts}}} - \underbrace{\varepsilon_0 E \cdot \frac{\partial \vec{E}}{\partial t}}_{= \frac{1}{2}\frac{\partial E^2}{\partial t}} \right]$$

$$\vec{\nabla}\cdot(\vec{E}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{B})$$

$$\frac{dW_m}{dt} = \int d^3r \left[\frac{1}{\mu_0}\vec{B}\cdot(\vec{\nabla}\times\vec{E}) - \frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B}) - \varepsilon_0 \frac{1}{2}\frac{\partial E^2}{\partial t} \right]$$

use $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ Faraday

use $\vec{B}\cdot\frac{\partial \vec{B}}{\partial t} = \frac{1}{2}\frac{\partial B^2}{\partial t}$

$$= \int d^3r \left[\left(-\frac{1}{2}\right)\left(\frac{\partial B^2}{\mu_0 \partial t} + \varepsilon_0 \frac{\partial E^2}{\partial t}\right) - \frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B}) \right]$$

$$\frac{dW_m}{dt} = -\frac{d}{dt}\int_{Vol} d^3r \left(\frac{1}{2\mu_0}B^2 + \frac{\varepsilon_0}{2}E^2\right) - \frac{1}{\mu_0}\oint_{Surface} d\vec{a}\cdot(\vec{E}\times\vec{B})$$

define $W_{EB} = \int_{Vol} d^3r \left(\frac{1}{2\mu_0}B^2 + \frac{\varepsilon_0}{2}E^2\right)$ electro-magneto energy in volume V

$\vec{S} \equiv \frac{\vec{E}\times\vec{B}}{\mu_0}$ "Poynting vector"

= energy density current

$$\frac{dW_m}{dt} = -\frac{d}{dt}W_{EB} - \oint d\vec{a}\cdot\vec{S}$$

(kinetic energy of charges)
increase in mechanical energy = energy lost from $\vec{E}+\vec{B}$ fields minus energy from $E+B$ fields flowing out of volume through surface

write $u_{EB} = \frac{\varepsilon_0}{2}E^2 + \frac{B^2}{2\mu_0}$ energy density of electromagnetic fields

u_M = mechanical energy density

$$\frac{d}{dt}\int_V d^3r\, u_M + \frac{d}{dt}\int_V d^3r\, u_{EB} = -\oint_S \vec{S}\cdot d\vec{a} = -\int_V d^3r\, \vec{\nabla}\cdot\vec{S}$$

$$\frac{\partial}{\partial t}\left(u_M + u_{EB}\right) = -\vec{\nabla}\cdot\vec{S}$$

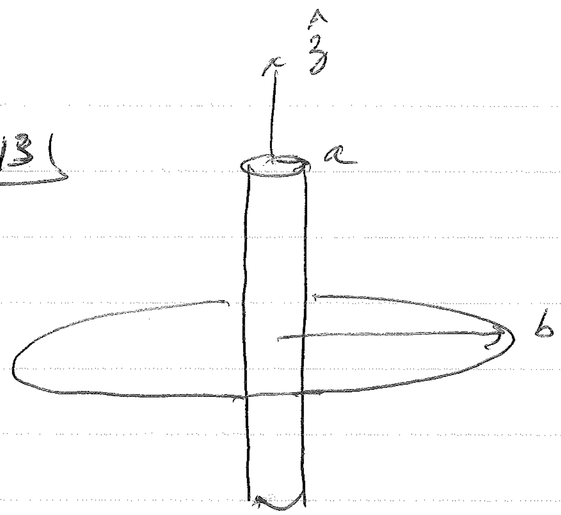
law of local conservation of energy for e-m fields

(same form as charge conservation $\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\vec{j}$

$\vec{S}$ is flux of energy carried by $\vec{E}+\vec{B}$ fields

$\oint_S \vec{S}\cdot d\vec{a}$ is energy per unit time carried by $\vec{E}+\vec{B}$ fields through surface S

8.13)



solenoid of radius a, surrounded by a circular wire ring of radius b
$a \ll b$

solenoid has current I_s flowing in N turns of wire per unit length

a) $\vec{B}$ field from solenoid is

$$\vec{B} = \begin{cases} \mu_0 N I_s \hat{z} & \text{for } r < a \\ 0 & \text{for } r > a \end{cases}$$

Flux through circular wire is

$$\Phi = \pi a^2 B = \mu_0 \pi a^2 N I_s$$

If Φ changes (because I_s changes) then there is an emf $\mathcal{E}$ induced in the circular wire ring

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\mu_0 \pi a^2 N \frac{dI_s}{dt}$$

If the circular wire ring has a total resistance R then the current induced in the ring is

$$\vec{I}_r = \frac{\mathcal{E}}{R}\hat{\varphi} = -\frac{\mu_0 \pi a^2 N}{R}\frac{dI_s}{dt}\hat{\varphi}$$

Note: Φ computed in $+\hat{z}$ direction $\Rightarrow$ $\mathcal{E}$ computed in the $+\hat{\varphi}$ (counterclockwise) direction (right hand rule)

If $\frac{dI_s}{dt} > 0$ then $\mathcal{E} < 0 \Rightarrow I_r$ is flowing in the $-\hat{\varphi}$ direction, ie flowing clockwise.

b) power dissipated in ring is $P = I_r^2 R = I_r \mathcal{E}$
where does this power come from? It must be coming from the solenoid!

Compute the flux of energy flowing away from the solenoid.

energy flux given by the Poynting vector

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

we want to evaluate this on the outside surface of the solenoid.

The $\vec{E}$ field in this $\vec{S}$ is just the $\vec{E}$ field induced by Faradays Law. By symmetry we expect
$\vec{E}(\vec{r}) = E(r)\hat{\varphi}$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int d\vec{a} \cdot \vec{B}$$

for a circular path of radius r we then get

$$2\pi r E(r) = -\frac{d\Phi}{dt} = \mathcal{E} \leftarrow \text{the emf in the ring}$$

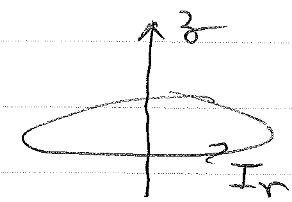
So we can write

$$\vec{E}(\vec{r}) = \frac{\mathcal{E}}{2\pi r}\hat{\varphi} \qquad \mathcal{E} = -\mu_0 \pi a^2 N \frac{dI_s}{dt} \text{ from (a)}$$

What is the $\vec{B}$ that appears in $\vec{S}$? It is NOT the $\vec{B}$ field from the solenoid as that is zero outside the solenoid. Rather the $\vec{B}$ outside the solenoid must be the $\vec{B}$ produced by the current I_r in the circular ring!

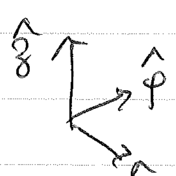
From example 5-6 in the text we have for $\vec{B}$ along the $\hat{z}$ from a circular ring

$$\vec{B}(z) = \frac{\mu_0 I_r}{2} \frac{b^2 \hat{z}}{(b^2+z^2)^{3/2}}$$



We really want $\vec{B}$ at $r=a$ on the surface of the solenoid, but since $a \ll b$ it will be a good enough approximation to use the above $\vec{B}$ at $r=0$ along the z axis.

So now $$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\underbrace{\left(\frac{\mathcal{E}}{2\pi a}\right)}_{E(r=a)}\underbrace{\left(\frac{\mu_0 I_r b^2}{2(b^2+z^2)^{3/2}}\right)}_{B}\hat{\varphi}\times\hat{z}$$



$\hat{\varphi}\times\hat{z} = \hat{r}$

$$\vec{S} = \frac{\mathcal{E} I_r b^2}{2\pi a} \frac{1}{2(b^2+z^2)^{3/2}}\hat{r}$$

on surface of solenoid.

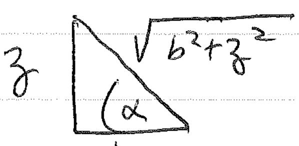
To get the total energy per unit time flowing away from the solenoid we integrate the flux of $\vec{S}$ through the surface of the solenoid.

$$P = \int d\vec{a} \cdot \vec{S} = \int_{-\infty}^{\infty} dz \int_0^{2\pi} d\varphi \, a \, \hat{r} \cdot \vec{S}$$

$$= \mathcal{E} I_r \frac{b^2}{2\pi a} \, 2\pi a \int_{-\infty}^{\infty} dz \frac{1}{2(b^2+z^2)^{3/2}}$$

$$= \mathcal{E} I_r \int_{-\infty}^{\infty} dz \frac{b^2}{2(b^2+z^2)^{3/2}}$$

to do the integral one makes a trig substitution



$$b \tan\alpha = z \Rightarrow dz = \frac{b}{\cos^2\alpha} d\alpha$$

$$\frac{1}{\sqrt{z^2+b^2}} = \frac{\cos\alpha}{b}$$

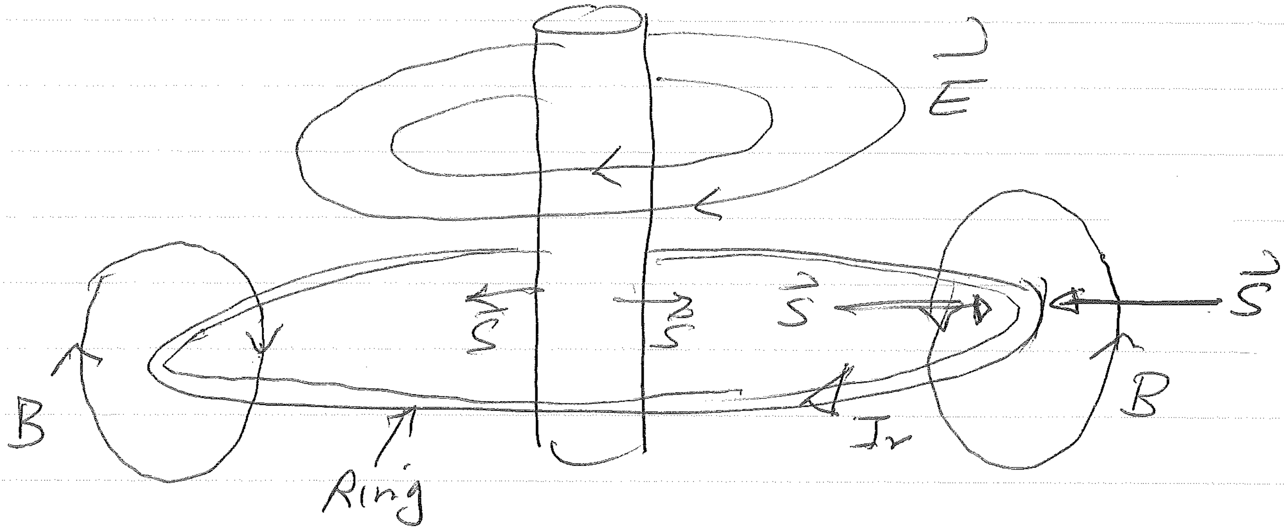
$$\int_{-\infty}^{\infty} dz \frac{1}{(b^2+z^2)^{3/2}} = \int_{-\pi/2}^{\pi/2} d\alpha \frac{b}{\cos^2\alpha} \frac{\cos^3\alpha}{b^3} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\alpha \frac{\cos\alpha}{b^2}$$

$$= \frac{2}{b^2}$$

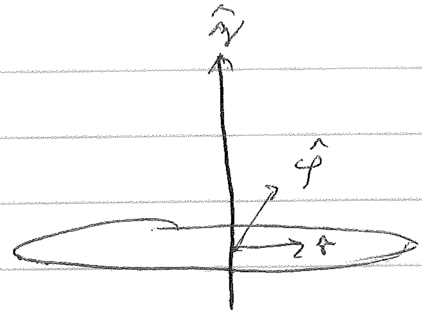
So $\displaystyle\int_{-\infty}^{\infty} dz \frac{b^2}{2(b^2+z^2)^{3/2}} = 1$ and $P = \mathcal{E} I_r$

power leaving solenoid = power dissipated in ring!

More generally



At surface of ring, one can see that $\vec{S}$ is always directed inward into the ring.



Biot - Savart Law

$$B(\vec r) = \frac{\mu_0}{4\pi}\int d^3r' \; \frac{\vec J(r') \times (\vec r - \vec r')}{|\vec r - \vec r'|^3}$$

$$B(z)\hat z = \frac{\mu_0 I_r}{4\pi}\oint d\vec\ell \times \frac{(z\hat z - b\hat r)}{(z^2+b^2)^{3/2}}$$

$\hat\varphi \times \hat z = \hat r$

$\hat\varphi \times \hat r = -\hat z$

$$= \frac{\mu_0 I_r}{4\pi}\oint_0^{2\pi} d\varphi \, b \, \frac{\hat\varphi \times (z\hat z - b\hat r)}{(z^2+b^2)^{3/2}}$$

$$= \frac{\mu_0 I_r}{4\pi}\frac{b}{(z^2+b^2)^{3/2}}\int_0^{2\pi} d\varphi \, (\underbrace{z\hat r}_{\text{integrates to zero}} + b\hat z)$$

$$\boxed{B(z)\hat z = \frac{\mu_0 I_r}{2}\frac{b^2}{(z^2+b^2)^{3/2}}}$$