Momentum Conservation

want similar conservation law for mechanical & electromagnetic momentum

$$\frac{\partial}{\partial t}\left(p_{mi} + p_{EBi} \right) = \vec{\nabla}\cdot\vec{T}_i \qquad i = x, y, z$$

p_{mi} = i^{th} component of mechanical momentum density

p_{EBi} = i^{th} component of electromagnetic momentum density

$\vec{T}_i$ = $-$flux density (or "current") of i^{th} component of momentum density

Since $\vec{T}_i$ is a vector with 3 components, and there are three such vectors, for $i = x, y, z$, we will see that these 3 vectors form the components of a 3×3 tensor (ie matrix)

mechanical momentum density given by Newtons Law $\Bigg\}$ $\dfrac{\partial \vec{p}_m}{\partial t} = \vec{f} = \rho\vec{E} + \vec{j}\times\vec{B}$

force density $\vec{f} = \rho\vec{E} + \vec{j}\times\vec{B} = \varepsilon_0(\vec{\nabla}\cdot\vec{E})\vec{E} + \left(\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B}$

Now apply vector algebra + Maxwell's equ (see text sec 8.2.2) to manipulate into the form

$$\vec{f} = \varepsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E}\right] + \frac{1}{\mu_0}\left[(\vec{\nabla}\cdot\vec{B})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{B}\right]$$
$$- \frac{1}{2}\vec{\nabla}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \varepsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) \qquad \cancel{(7.91)}\ (8.15)$$

$\vec{f}$ above looks like mess, but it simplifies if one introduces the following 3x3 matrix, known as the "Maxwell Stress Tensor"

$$T_{ij} = \varepsilon_0 \left(E_i E_j - \tfrac{1}{2} \delta_{ij} E^2 \right) + \tfrac{1}{\mu_0} \left(B_i B_j - \tfrac{1}{2} \delta_{ij} B^2 \right)$$

$$i = x, y, z \quad \text{and} \quad \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$\overleftrightarrow{T} = \begin{bmatrix} \varepsilon_0 (E_x^2 - \tfrac{1}{2}E^2) + \tfrac{1}{\mu_0}(B_x^2 - \tfrac{1}{2}B^2) & \varepsilon_0 E_x E_y + \tfrac{1}{\mu_0} B_x B_y & \varepsilon_0 E_x E_z + \tfrac{1}{\mu_0} B_x B_z \\ \varepsilon_0 E_x E_y + \tfrac{1}{\mu_0} B_x B_y & \varepsilon_0 (E_y^2 - \tfrac{1}{2}E^2) + \tfrac{1}{\mu_0}(B_y^2 - \tfrac{1}{2}B^2) & \varepsilon_0 E_y E_z + \tfrac{1}{\mu_0} B_y B_z \\ \varepsilon_0 E_x E_z + \tfrac{1}{\mu_0} B_x B_z & E_y E_z + \tfrac{1}{\mu_0} B_y B_z & \varepsilon_0 (E_z^2 - \tfrac{1}{2}E^2) + \tfrac{1}{\mu_0}(B_z^2 - \tfrac{1}{2}B^2) \end{bmatrix}$$

$T_{ij} = T_{ji} \Rightarrow T$ is symmetric

$$(\vec{\nabla} \cdot \overleftrightarrow{T}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

$$(\vec{\nabla} \cdot \overleftrightarrow{T})_j = \sum_i \frac{\partial}{\partial x_i} T_{ij} = \sum_i \Big[\varepsilon_0 \left(\frac{\partial E_i}{\partial x_i} E_j + E_i \frac{\partial E_j}{\partial x_i} - \frac{1}{2} \frac{\partial E^2}{\partial x_i} \delta_{ij} \right)$$
$$+ \frac{1}{\mu_0} \left(\frac{\partial B_i}{\partial x_i} B_j + B_i \frac{\partial B_j}{\partial x_i} - \frac{1}{2} \frac{\partial B^2}{\partial x_i} \delta_{ij} \right) \Big]$$

$$(\vec{\nabla} \cdot \overleftrightarrow{T}) = \varepsilon_0 \left[(\vec{\nabla} \cdot \vec{E}) \vec{E} + (\vec{E} \cdot \vec{\nabla}) \vec{E} - \tfrac{1}{2} \vec{\nabla} E^2 \right]$$
$$+ \frac{1}{\mu_0} \left[(\vec{\nabla} \cdot \vec{B}) \vec{B} + (\vec{B} \cdot \vec{\nabla}) \vec{B} - \tfrac{1}{2} \vec{\nabla} B^2 \right]$$
$$= \vec{f} + \varepsilon_0 \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) = \vec{f} + \varepsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t}$$

$$\vec{f} = \frac{\partial \vec{p}_M}{\partial t} = -\varepsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t} + \vec{\nabla} \cdot \overleftrightarrow{T}$$

$$\frac{\partial}{\partial t}\left[\vec{p}_M + \varepsilon_0 \mu_0 \vec{S}\right] = \vec{\nabla} \cdot \overleftrightarrow{T}$$ this is the desired conservation law for momentum

$\Rightarrow \boxed{\varepsilon_0 \mu_0 \vec{S} = \vec{p}_{EM}}$ electromagnetic momentum density

~~$-\overleftrightarrow{T}$ = current or flux~~

$-\overleftrightarrow{T}_{ij}$ is i^{th} component of current, of j^{th} component of electromagnetic momentum density

ie the vector $-\begin{pmatrix} T_{xx} \\ T_{yx} \\ T_{zx} \end{pmatrix}$ is the current for x-component $\vec{p}_{EMx}$ of E-M momentum

integral form: $\int_{Vol} d^3r \left[\frac{\partial}{\partial t} \vec{p}_M + \frac{\partial}{\partial t} \vec{p}_{EB}\right] = \frac{d}{dt} \int_{Vol} d^3r \underbrace{\left(\vec{p}_m + \vec{p}_{EB}\right)}_{\text{total mechanical + electromagnetic field momentum contained in Vol}}$

$$= \int_{Vol} d^3r \, \vec{\nabla} \cdot \overleftrightarrow{T} = \underbrace{\oint_S d\vec{a} \cdot \overleftrightarrow{T}}_{\text{(−) flux of field momentum out through surface } S \text{ bounding Vol}}$$

Or we can write

$$\frac{d}{dt}\int_{vol} d^3r\ \vec{p}_{mech} = \frac{d\vec{P}_{mech}}{dt} = -\frac{d}{dt}\int_{vol} d^3r\ \vec{p}_{ER} + \oint_S d\vec{a}\cdot\overleftrightarrow{T}$$

$$\frac{d\vec{P}_{mech}}{dt} = \underbrace{-\frac{d\vec{P}_{EB}}{dt} + \oint_S d\vec{a}\cdot\overleftrightarrow{T}}_{\text{total electromagnetic force on the volume}}$$

$$\vec{F}_{EB} = \frac{d\vec{P}_{mech}}{dt}$$

For a situation where $\vec{E}$ and $\vec{B}$ are constant in time,

$$\frac{d\vec{P}_{EB}}{dt} = 0 \quad \text{and so}$$

when doing the surface integral, $d\vec{a} = da\,\hat{n}$, the normal $\hat{n}$ points <u>outwards</u> of vol

$$\frac{d\vec{P}_{mech}}{dt} = \oint_S d\vec{a}\cdot\overleftrightarrow{T}$$ ← gives total electromagnetic force on the volume

This is why $\overleftrightarrow{T}$ is called the Maxwell <u>stress</u> tensor

It is like a pressure acting on the walls of the volume.

Numerically compute $\varepsilon_0\mu_0$, find $\varepsilon_0\mu_0 = \frac{1}{c^2}$ with c = speed of light

$\vec{S} = c^2 \vec{p}_{EB}$

↑ momentum density

↑ energy current

Suppose energy current is made of "particles" that travel with velocity $\vec{c}$. Then $\vec{S} = \vec{c}\, u_{EB}$ $\quad u_{EB}$ is energy density

$u_{EB} = c\, p_{EB}$ = energy-moment relation for photons.

Also can do same for angular momentum

$$\vec{\mathcal{L}}_{EB} = \vec{r} \times \vec{p}_{EB} = \varepsilon_0 \vec{r} \times (\vec{E} \times \vec{B})$$

= angular momentum density contained in $\vec{E}$ and $\vec{B}$ fields

see Griffiths Sec 8.2.4

Sec 7.3.4

Magnetic monopoles

Maxwell's equations: $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ $\qquad \vec{\nabla}\cdot\vec{B} = 0$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

Maxwell's equations would have a much more symmetric look to them, if one imagined that there were such things as magnetic monopoles (ie. magnetic charges)

$\vec{\nabla}\cdot\vec{B} = 0$ is purely expt result. Suppose we found a magnetic monopole, so that we would now have

$$\vec{\nabla}\cdot\vec{B} = \mu_0\eta$$

η = volume density of magnetic charge.

$\eta = \sum_i g_i\,\delta(\vec{r}-\vec{r}_i)$ for point monopoles

A point magnetic monopole g would produce a magnetic field $\vec{B} = \frac{\mu_0}{4\pi}\frac{g}{r^2}\hat{r}$

There would be conservation law of magnetic charge $\vec{\nabla}\cdot\vec{k} = -\frac{\partial\eta}{\partial t}$ where $\vec{k}$ is the magnetic charge current density.

Then Faraday's Law would have to be fixed, like Maxwell fixed Ampere's Law

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t}\right) = 0 + \frac{\partial\,\vec{\nabla}\cdot\vec{B}}{\partial t} = \mu_0\frac{\partial\eta}{\partial t} = -\mu_0\vec{\nabla}\cdot\vec{k}$$

New Faradays law would be $\vec{\nabla}\times\vec{E} = -\mu_0\vec{k} - \frac{\partial\vec{B}}{\partial t}$

Now Maxwells equations would look symmetric!

$$\left.\begin{aligned}&\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \qquad &&\vec{\nabla}\cdot\vec{B} = \mu_0\eta\\ &\vec{\nabla}\times\vec{E} = -\mu_0\vec{k} - \frac{\partial\vec{B}}{\partial t} &&\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\end{aligned}\right.$$

Despite the elegant appearence of Maxwell's Eqn if we allow magnetic monopoles, no experiment has ever convincingly found them.
Nevertheless Dirac gave a very interesting theoretical arguement to show that if monopoles existed, one could explain why charge is quantized!

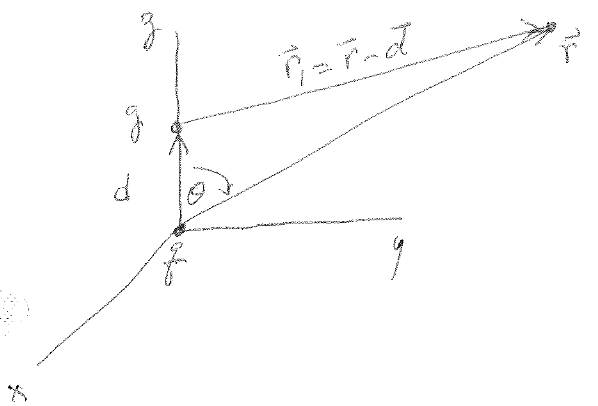
Griffiths ~~7.62~~ 8.19 – Dirac's arguement of why electric charge is quantized.

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3}$$

$\vec{E}$ from pt electric charge at the origin $\vec{r}=0$

$$\vec{B} = \frac{\mu_0 g}{4\pi}\frac{\hat{r}_1}{r_1^2} = \frac{\mu_0 g}{4\pi}\frac{\vec{r}_1}{r_1^3} = \frac{\mu_0 g}{4\pi}\frac{\vec{r}-\vec{d}}{|\vec{r}-\vec{d}|^3} \qquad \vec{r}_1 = \vec{r}-\vec{d}$$

$\vec{B}$ from pt magnetic monopole at $\vec{r}=\vec{d}$



angular momentum density

$$\vec{\mathcal{L}}_{EB} = \vec{r}\times\vec{p}_{EB} = \varepsilon_0\,\vec{r}\times(\vec{E}\times\vec{B})$$

$$= \varepsilon_0\frac{\mu_0 g}{4\pi}\frac{q}{4\pi\varepsilon_0}\frac{1}{r^3}\frac{1}{|\vec{r}-\vec{d}|^3}\,\vec{r}\times(\vec{r}\times(\vec{r}-\vec{d}))$$

$$= -\frac{\mu_0 g q}{(4\pi)^2}\frac{1}{r^3|\vec{r}-\vec{d}|^3}\,\vec{r}\times(\vec{r}\times\vec{d})$$

$$\vec{r}\times(\vec{r}\times\vec{d}) = \vec{r}(\vec{r}\cdot\vec{d}) - \vec{d}(\vec{r}\cdot\vec{r}) = \vec{r}\,rd\cos\theta - \vec{d}r^2$$

alternatively, $\vec{r}\times\vec{d} = -rd\sin\theta\,\hat{\phi}$

$\vec{r}\times(\vec{r}\times\vec{d}) = r^2 d\sin\theta\,\hat{\theta}$

($\vec{p}_{EM}$ is in $\hat{\phi}$ direction)

By rotation symmetry about $\hat{z}$ axis, total angular momentum $\vec{L}_{EB} = \int d^3r\, \vec{\mathcal{L}}_{EB}$ can have only non zero component along $\hat{z}$.

$$\hat{z}\cdot(\vec{r}\times(\vec{r}\times\vec{d})) = (\hat{z}\cdot\vec{r})\,rd\cos\theta - (\hat{z}\cdot\vec{d})r^2$$
$$= (r\cos\theta)\,rd\cos\theta - dr^2$$
$$= r^2 d\cos^2\theta - r^2 d = -r^2 d\sin^2\theta$$

$$\hat{z}\cdot\vec{\mathcal{L}}_{EB} = \frac{\mu_0 g q}{(4\pi)^2}\frac{r^2 d\sin^2\theta}{r^3|\vec{r}-\vec{d}|^3}$$

$$|\vec r - \vec d|^3 = (r^2+d^2-2rd\cos\theta)^{3/2}$$

$$\hat z\cdot \vec L_{EB} = \int d^3r\ \hat z\cdot \vec \ell_{EB} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dr\, r^2\ \frac{\mu_0 g q}{(4\pi)^2}\ \frac{r^2 d\sin^2\theta}{r^3(r^2+d^2-2rd\cos\theta)^{3/2}}$$

$$= 2\pi\ \frac{\mu_0 g q d}{(4\pi)^2}\int_0^{\pi} d\theta \int_0^{\infty} dr\ \frac{r\sin^3\theta}{(r^2+d^2-2rd\cos\theta)^{3/2}}$$

look up in integral tables

$$\int dx\ \frac{x}{(ax^2+bx+c)^{3/2}} = \frac{2(bx+2c)}{(b^2-4ac)\sqrt{ax^2+bx+c}}$$

apply to above with $a\equiv 1$, $b\equiv -2d\cos\theta$, $c\equiv d^2$, $x\equiv r$

$$\int_0^{\infty} dr\ \frac{r}{(r^2+d^2-2rd\cos\theta)^{3/2}} = \left[\frac{-4d\cos\theta\, r+4d^2}{(4d^2\cos^2\theta-4d^2)\sqrt{r^2+d^2-2rd\cos\theta}}\right]_{r=0}^{\infty}$$

$$= \frac{-4d\cos\theta}{4d^2\cos^2\theta-4d^2} - \frac{4d^2}{(4d^2\cos^2\theta-4d^2)\,d}$$

$$\cancel{+\frac{4d^2\cos\theta-4d^2}{(4d^2\cos^2\theta-4d^2)\,d}}$$

$$= \frac{-\cos\theta}{d(\cos^2\theta-1)} - \frac{1}{d(\cos^2\theta-1)} = \frac{1+\cos\theta}{d(1-\cos^2\theta)} = \frac{1+\cos\theta}{d(1+\cos\theta)(1-\cos\theta)}$$

$$= \frac{1}{d(1-\cos\theta)}$$

$$\hat z\cdot\vec L_{EB} = \frac{\mu_0 g q d}{8\pi d}\int_0^{\pi} d\theta\ \frac{\sin^3\theta}{1-\cos\theta}$$

use $x=-\cos\theta$
$dx = d\theta\sin\theta$
$\sin^2\theta = 1-x^2$

$$= \frac{\mu_0 g q}{8\pi}\int_{-1}^{1} dx\ \frac{1-x^2}{1+x} = \frac{\mu_0 g q}{8\pi}\int_{-1}^{1} dx\ (1-x)$$

$$\hat{z}\cdot\vec{L}_{EB} = \frac{\mu_0 g q}{8\pi}\left(x-\frac{x^2}{2}\right)\Big|_{-1}^{1} = \frac{\mu_0 g q}{8\pi}\left[\left(1-\frac{1}{2}\right)-\left(-1-\frac{1}{2}\right)\right] = \frac{\mu_0 g q}{4\pi}$$

total angular momentum in charge - monopole pair

$$\vec{L}_{EB} = \frac{\mu_0 g q}{4\pi}\hat{z}$$

Note, $\vec{L}_{EB}$ doesnt depend on distance d between charge and monopole!

In quantum mechanics one learns that angular momentum must be quantized

$$L_z = \left(\frac{n}{2}\right)\hbar$$

where n is always an integer
$\hbar$ is Planck's constant
(n even for Boson
n odd for Fermion)

apply this quantization to the field angular momentum of the charge - monopole pair and one gets

$$\frac{\mu_0 g q}{4\pi} = \frac{n}{2}\hbar$$

$$\boxed{g q = 2\pi n \frac{\hbar}{\mu_0}}$$

The product gq can only be quantized in units of $\frac{2\pi\hbar}{\mu_0} = \frac{h}{\mu_0}$, if electric charge q is quantized in units q_0 (ie. $q = m q_0$, m integer) and g is quantized in units g_0 (ie $g = m' g_0$, m' integer) where $g_0 q_0 = \frac{h}{\mu_0}$

Then $g q = m m' \frac{h}{\mu_0} = n \frac{h}{\mu_0}$ where n is always integer!