Maxwell's Equations in Potential Form

1) $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$ the same as in magnetostatics
$\vec{A}$ is the "vector potential"

2) $\vec{\nabla}\times\vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ Since $\vec{\nabla}\times\vec{E} \neq 0$ in general, we cannot write $\vec{E} = -\vec{\nabla}V$ as we did for statics.

Instead

$$\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) \Rightarrow \vec{\nabla}\times\left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

$$\Rightarrow \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}V \quad \text{or} \quad \boxed{\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}}$$

V is the "scaler potential"

3) $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \Rightarrow \vec{\nabla}\cdot\left(-\vec{\nabla}V - \dfrac{\partial \vec{A}}{\partial t}\right) = \rho/\varepsilon_0$

$$\text{or} \quad \boxed{\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\rho/\varepsilon_0} \quad (*)$$

4) $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\dfrac{\partial \vec{E}}{\partial t} \Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J} - \mu_0\varepsilon_0\left(\vec{\nabla}\dfrac{\partial V}{\partial t} + \dfrac{\partial^2\vec{A}}{\partial t^2}\right)$

use $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = -\nabla^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A})$ to rewrite as

$$\boxed{\left(\nabla^2\vec{A} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right) - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu_0\varepsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\vec{J}} \quad (**)$$

Gauge Transformations

The V and $\vec{A}$ that give a particular $\vec{E}$ and $\vec{B}$ are not unique

Recall statics $\vec{E} = -\vec{\nabla} V \Rightarrow V' = V + c$ with C a constant gives the same $\vec{E}$

$$-\vec{\nabla} V' = -\vec{\nabla} V - \vec{\nabla} C = -\vec{\nabla} V + 0 = \vec{E}$$

$\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \vec{A}' = \vec{A} + \vec{\nabla}\lambda$ for any scalar function $\lambda(\vec{r})$ gives the same $\vec{B}$

$$\vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times (\vec{\nabla}\lambda) = \vec{\nabla} \times \vec{A} + 0 = \vec{B}$$

Back to dynamics

$\vec{A}' = \vec{A} + \vec{\nabla}\lambda$ still gives the same $\vec{B}$ as does $\vec{A}$

But now since $\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$, if we change from $\vec{A} \rightarrow \vec{A}'$ then we would be changing $\vec{E}$, UNLESS we make some corresponding change in V.

For $\vec{A} = \vec{A}' - \vec{\nabla}\lambda$, $\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V - \frac{\partial \vec{A}'}{\partial t} + \vec{\nabla}\frac{\partial \lambda}{\partial t}$

so

$$\vec{E} = -\vec{\nabla}\left(V - \frac{\partial \lambda}{\partial t}\right) - \frac{\partial \vec{A}'}{\partial t}$$

so $\vec{E}$ will not change if we let $V' = V - \frac{\partial \lambda}{\partial t}$

So

$$\boxed{\vec{A} \rightarrow \vec{A} + \vec{\nabla}\lambda = \vec{A}' \quad \text{AND} \quad V \rightarrow V - \frac{\partial \lambda}{\partial t} = V'}$$

leave the fields $\vec{E}$ and $\vec{B}$ unchanged

The transformation $\vec{A}' = \vec{A} + \vec{\nabla}\lambda$, $V' = V - \frac{\partial \lambda}{\partial t}$ is called a gauge transformation

for any scalar function $\lambda(\vec{r},t)$, the gauge transformation leaves $\vec{B}$ and $\vec{E}$ unchanged

$$\vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times\vec{A}' \;, \quad \vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}V' - \frac{\partial \vec{A}'}{\partial t}$$

We can therefore use the freedom, given by the arbitrary $\lambda(\vec{r},t)$, to impose some additional constraint on the potentials, in particular to make $\vec{\nabla}\cdot\vec{A}$ equal to some desired quantity which will simplify the equations (*) and (**). Making such a choice for $\vec{\nabla}\cdot\vec{A}$ is called "fixing the gauge".

i) Coulomb gauge : same as used in magnetostatics

require $\vec{\nabla}\cdot\vec{A} = 0$

How do we know we can always find such an $\vec{A}$? suppose we have an $\vec{A}$ such that $\vec{B} = \vec{\nabla}\times\vec{A}$ but $\vec{\nabla}\cdot\vec{A} \neq 0$. Then define $\vec{A}' = \vec{A} + \vec{\nabla}\lambda$. We want

$\vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\vec{A} + \nabla^2\lambda = 0$. This will be satisfied if we can find a $\lambda(\vec{r},t)$ that is a solution to

$$-\nabla^2\lambda = \underbrace{\vec{\nabla}\cdot\vec{A}}_{\substack{\text{this is a known} \\ \text{function of } \vec{r},t \text{ since we know } \vec{A}}}$$

This is just Poisson's equation!

For the case where sources are localized and our system is all of space out to infinity, the solution is

$$\lambda(\vec{r},t) = \frac{1}{4\pi}\int d^3r' \, \frac{\vec{\nabla}'\cdot\vec{A}(\vec{r}',t)}{|\vec{r}-\vec{r}'|}$$

so since we can always find the needed λ, we can always transform from $\vec{A}$ to a new $\vec{A}'$ with $\vec{\nabla}\cdot\vec{A}'=0$

In the Coulomb gauge with $\vec{\nabla}\cdot\vec{A}=0$

$(*) \Rightarrow \nabla^2 V = -\rho/\varepsilon_0$ Poisson's equation just like in statics!

Solution is $V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \, \frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}$

The above looks troubling because it says that V at position $\vec{r}$ at time t, is determined by all the charges at different $\vec{r}'$ at exactly the same time t! This implies action at a distance - the charge at $\vec{r}'$ effects the potential some distance away at $\vec{r}$ instantaneously. But we believe that information cannot travel instantaneously fast, it can travel no faster than the speed of light.

The solution to this paradox is to realize that in dynamics, $\vec{E}$ is not given solely by V, but rather $\vec{E} = -\vec{\nabla} V - \frac{\partial\vec{A}}{\partial t}$ depends also on $\vec{A}$

So the instantaneous relation between ρ and V can be canceled out by the term involving $\vec{A}$, and the relation between ρ and $\vec{E}$ should wind up causal (i.e. it takes a finite time for a charge at $\vec{r}'$ to affect the $\vec{E}$ at $\vec{r}$, and the transmission of this effect travels with the speed of light)

$$(**) \Rightarrow \nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} + \mu_0 \varepsilon_0 \vec{\nabla}\left(\frac{\partial V}{\partial t}\right)$$

use solution for V in terms of ρ for the Coulomb gauge to get

$$\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} + \frac{\mu_0 \varepsilon_0}{4\pi \varepsilon_0} \vec{\nabla} \int d^3 r' \left(\frac{\partial \rho(\vec{r}', t)}{\partial t}\right) \frac{1}{|\vec{r} - \vec{r}'|}$$

use $\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J}$ to get

$$\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} + \frac{\mu_0}{4\pi} \vec{\nabla} \int d^3 r' \frac{\left[-\vec{\nabla}' \cdot \vec{J}(\vec{r}', t)\right]}{|\vec{r} - \vec{r}'|}$$

relation between $\vec{A}$ and $\vec{J}$ is non-local in space
it is an "integro-differential" equation

Corollary to Helmholtz Theorem   Griffiths Appendix B

Any function $\vec{F}(\vec{r})$ that $\to 0$ sufficiently fast as $r \to \infty$ can always be written as

$$\vec{F}(\vec{r}) = \vec{F}_L(\vec{r}) + \vec{F}_T(\vec{r}) \quad \text{where } \vec{F}_L = -\vec{\nabla} U$$
$$\text{and } \vec{F}_T = \vec{\nabla} \times \vec{W}$$

$\vec{F}_L$ is called longitudinal part or curlfree part since $\vec{\nabla} \times \vec{F}_L = -\vec{\nabla} \times \vec{\nabla} U = 0$

$\vec{F}_T$ is called the transverse part or divergenceless part since $\vec{\nabla} \cdot \vec{F}_T = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{W}) = 0$

Moreover $$U(\vec{r}) = \frac{1}{4\pi} \int d^3r' \, \frac{\vec{\nabla}' \cdot \vec{F}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

and $$\vec{W}(\vec{r}) = \frac{1}{4\pi} \int d^3r' \, \frac{\vec{\nabla}' \times \vec{F}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

A consequence, if we know a function's divergence and curl, ie we know $\vec{\nabla} \cdot \vec{F}(\vec{r})$ and $\vec{\nabla} \times \vec{F}(\vec{r})$, then that is enough to compute U and $\vec{W}$ above and so construct the function $$\vec{F} = -\vec{\nabla} U + \vec{\nabla} \times \vec{W}$$

Knowing the divergence and curl of a vector function allows one to solve for the function!

Proof: Assume $\vec{F} = -\vec{\nabla}U + \vec{\nabla}\times\vec{W}$

then $\vec{\nabla}\cdot\vec{F} = -\nabla^2 U + \underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{W})}_{=0}$

so $-\nabla^2 U = \vec{\nabla}\cdot\vec{F}$

this is just Poisson's Equation and so has solution

$$U(\vec{r}) = \frac{1}{4\pi}\int d^3r' \, \frac{\vec{\nabla}'\cdot\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

Next: $\vec{\nabla}\times\vec{F} = \underbrace{-\vec{\nabla}\times(\vec{\nabla}U)}_{=0} + \underbrace{\vec{\nabla}\times(\vec{\nabla}\times\vec{W})}_{\vec{\nabla}(\vec{\nabla}\cdot\vec{W}) - \nabla^2\vec{W}}$

we can always transform $\vec{W}' = \vec{W} + \vec{\nabla}\lambda$ for any scalar $\lambda(\vec{r})$ without changing $\vec{F}$, so use this freedom to choose a $\vec{W}$ that obeys $\vec{\nabla}\cdot\vec{W} = 0$. Then

$$-\nabla^2\vec{W} = \vec{\nabla}\times\vec{F}$$

This is again just Poisson's Equation and so has solution

$$\vec{W}(\vec{r}) = \frac{1}{4\pi}\int d^3r' \, \frac{\vec{\nabla}'\times\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

So $\vec{F} = -\vec{\nabla}U + \vec{\nabla}W$ with U and $\vec{W}$ as above gives an function $\vec{F}$ with the correct divergence and curl.

But specify the divergence and curl of a function uniquely determines the function if one assumes $\vec{F}(\vec{r}) \to 0$ as $r \to \infty$

proof:

Assume $\vec{F}$ and $\vec{F}'$ both have the same divergence and curl

$\vec{\nabla}\cdot\vec{F} = \vec{\nabla}\cdot\vec{F}'$ and $\vec{\nabla}\times\vec{F} = \vec{\nabla}\times\vec{F}'$

Construct $\vec{G} = \vec{F} - \vec{F}'$

then $\vec{\nabla}\cdot\vec{G} = 0$ and $\vec{\nabla}\times\vec{G} = 0$

But the only such $\vec{G}$ that has both vanishing divergence and curl is a constant $\vec{G}_0$. So if we require that $\vec{F}(\vec{r}), \vec{F}'(\vec{r}) \to 0$ as $r \to \infty$, then we must have $\vec{F} = \vec{F}'$

Compare to (**) in the Coulomb gauge and we can write

$$\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} + \mu_0 \vec{J}_L$$

$$= -\mu_0 \vec{J}_T \quad \text{since } \vec{J} = \vec{J}_L + \vec{J}_T$$

$$\boxed{\Box^2 \vec{A} = -\mu_0 \vec{J}_T}$$

vector potential determined by the <u>transverse</u> part of the current

$$\Box^2 \equiv \nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}$$

Note: $\vec{J}_T(\vec{r})$ is not <u>locally</u> related to $\vec{J}(\vec{r})$

For dynamic problems, a more convenient gauge to work in is the

2) <u>Lorentz Gauge</u> where we require

$$\vec{\nabla} \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} = 0$$

We can always find potentials $\vec{A}$ and V that satisfy this condition.

<u>Proof</u>: Suppose $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$ but $\vec{\nabla} \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} \neq 0$. Then transform to

$$\vec{A}' = \vec{A} + \vec{\nabla} \lambda, \quad V' = V - \frac{\partial \lambda}{\partial t}$$

Then

$$\vec{\nabla} \cdot \vec{A}' + \mu_0 \varepsilon_0 \frac{\partial V'}{\partial t} = \vec{\nabla} \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} + \nabla^2 \lambda - \mu_0 \varepsilon_0 \frac{\partial^2 \lambda}{\partial t^2}$$

we can make this vanish if we can find a $\lambda(\vec{r},t)$ that satisfies

$$\underbrace{-\nabla^2\lambda + \mu_0\varepsilon_0 \frac{\partial^2\lambda}{\partial t^2}}_{-\Box^2\lambda} = \left[\vec{\nabla}\cdot\vec{A} + \mu_0\varepsilon_0 \frac{\partial V}{\partial t}\right]$$

known function of $\vec{r}, t$ since we know $\vec{A}$ and V

just like Poisson's eqn $-\nabla^2\lambda = g$ always has a solution, one can show that the inhomogeneous wave eqn $-\Box^2\lambda = g$ always has a solution

$\Rightarrow$ we can always find a $\lambda(\vec{r},t)$ that lets us transform to $\vec{A}'$ and V' that are in the Lorentz gauge

In the Lorentz gauge $\vec{\nabla}\cdot\vec{A} = -\mu_0\varepsilon_0 \frac{\partial V}{\partial t}$

and so

$$(*) \Rightarrow \nabla^2 V - \mu_0\varepsilon_0 \frac{\partial^2 V}{\partial t^2} = -\rho/\varepsilon_0$$

$$(**) \Rightarrow \nabla^2\vec{A} - \mu_0\varepsilon_0 \frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}$$

equations for V and $\vec{A}$ now both have the same simple form

$$\Box^2 \equiv \nabla^2 - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}$$

is D'Alembertian operator

wave equation operator

wave velocity $v^2 = \frac{1}{\mu_0\varepsilon_0}$

wave equation $\begin{cases} \Box^2 V = -\rho/\varepsilon_0 \\ \Box^2\vec{A} = -\mu_0\vec{J} \end{cases}$

hence forth we will use the Lorentz gauge for all non-static problems.

Note: If we are in a static situation where

$\frac{\partial V}{\partial t} = 0$, $\frac{\partial \vec{A}}{\partial t} = 0$, then

$\vec{\nabla} \cdot \vec{A} = -\mu_0 \varepsilon_0 \frac{\partial V}{\partial t} = 0$ Lorentz gauge → Coulomb gauge

$\Box^2 \rightarrow \nabla^2$

So

$\Box^2 V \rightarrow \nabla^2 V = -\rho/\varepsilon_0$
$\Box^2 \vec{A} \rightarrow \nabla^2 \vec{A} = -\mu_0 \vec{J}$

} just the familiar Poisson equations found for statics in the Coulomb gauge

Lorentz Force

for a particle with charge q moving on trajectory $\vec{r}(t)$

$$\vec{F} = \frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B}) = q\left(-\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} + \vec{v} \times (\vec{\nabla} \times \vec{A})\right)$$

$$= -q\left(\vec{\nabla} V + \frac{\partial \vec{A}}{\partial t}\right) - q\left((\vec{v} \cdot \vec{\nabla})\vec{A} - \vec{\nabla}(\vec{v} \cdot \vec{A})\right)$$

$$= -q\left(\frac{\partial \vec{A}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{A} + \vec{\nabla}(V - \vec{v} \cdot \vec{A})\right)$$

$\frac{\partial \vec{A}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{A} = \frac{d\vec{A}}{dt}$ convective derivative

$$\frac{d}{dt}\left(\vec{A}(\vec{r}(t), t)\right) = \frac{\partial \vec{A}}{\partial t} + \frac{\partial \vec{A}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{A}}{\partial y}\frac{dy}{dt} + \frac{\partial \vec{A}}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial \vec{A}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{A}$$

change in $\vec{A}$ as seen by a particle moving with velocity $\vec{v}$

$$\Rightarrow \frac{d\vec{p}}{dt} = -q\left(\frac{d\vec{A}}{dt} + \vec{\nabla}(V - \vec{v}\cdot\vec{A})\right)$$

$$\frac{d}{dt}\big(\underbrace{\vec{p} + q\vec{A}}_{\substack{\text{"canonical" momentum} \\ \vec{p}_{can}}}\big) = -\vec{\nabla}\big(\underbrace{qV - q\vec{v}\cdot\vec{A}}_{\text{"potential" } U}\big)$$

$$\frac{d\vec{p}_{can}}{dt} = -\vec{\nabla}U$$