Lagrangian Mechanics

generalized coordinates x_i and velocities $\dot{x}_i$

Lagrangian $\mathcal{L}(x_i, \dot{x}_i) = K - U$

kinetic energy — potential energy

Lagrange's equation of motion

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}$$

To make connection to Newtonian mechanics, identify

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = p_x \quad \text{canonical momentum}, \qquad \frac{\partial \mathcal{L}}{\partial x} = F_x \quad \text{force}$$

For a particle in a conservative force field given by the potential $U(\vec{r})$

$$\mathcal{L}(\vec{r}, \dot{\vec{r}}) = \frac{1}{2}m|\dot{\vec{r}}|^2 - U(\vec{r})$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} = m\dot{\vec{r}} = \vec{p} \quad \text{mechanical momentum}$$

$$\frac{\partial \mathcal{L}}{\partial \vec{r}} = -\vec{\nabla} U = \vec{F}$$

Lagrange equation gives $\frac{d\vec{p}}{dt} = \vec{F}$

How do we describe a charged particle in $\vec{E}$ and $\vec{B}$ fields?

Use Lagrangian with $U(\vec{r},\dot{\vec{r}}) = qV - q\dot{\vec{r}}\cdot\vec{A}$

scalar potential vector potential

$$\mathcal{L} = \frac{1}{2}m(\dot{\vec{r}})^2 - qV + q\dot{\vec{r}}\cdot\vec{A}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}} = m\dot{\vec{r}} + q\vec{A} = m\vec{v} + q\vec{A} = \boxed{\vec{p}_{canonical} = \vec{p} + q\vec{A}}$$

canonical momentum $\neq$ mechanical momentum

$$\frac{\partial\mathcal{L}}{\partial\vec{r}} = -\vec{\nabla}U = -q\vec{\nabla}V + q\vec{\nabla}(\dot{\vec{r}}\cdot\vec{A}(\vec{r}(t),t))$$

Lagrange Equation then gives:

$$\frac{d\vec{p}_{canonical}}{dt} = -\vec{\nabla}(qV - q\vec{v}\cdot\vec{A})$$

what we derived at end of last lecture from Newtons equation and Lorentz force!

How does this go over in quantum mechanics?

Consider a charge q in static $\vec{E}$ and $\vec{B}$ fields

For quantum mechanics we need to construct the Hamiltonian operator

$$\mathcal{H} = \underbrace{\frac{\vec{p}^2}{2m}}_{\text{kinetic}} + \underbrace{U}_{\text{potential}} \quad \text{then} \quad \mathcal{H}\psi = E\psi$$

gives eigenwavefunctions and eigenvalues

take $\vec{p} \to -i\hbar\vec{\nabla}$

$$\vec{p}^2 \to -\hbar^2\nabla^2$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi + U\psi = 0 \quad \text{Schrodinger Eqn!}$$

for static $\vec{E}$ and $\vec{B}$, U is just qV with V the electrostatic potential $\vec{E} = -\vec{\nabla}V$

But if there is a static $\vec{B}$, how does it appear in the Hamiltonian? There is no potential energy associated with $\vec{B}$. If the Schrodinger equation as above still holds with no modifications, the the eigenstates of a charge in a static $\vec{B}$ would be the same as the eigenstates when $\vec{B} = 0$!

Solution is that identification $\vec{p} \to -i\hbar\vec{\nabla}$ does not apply to the mechanical momentum $\vec{p} = m\vec{v}$, but rather to the canonical momentum $\vec{p}_{can}$

$$\mathcal{H} = \frac{1}{2m} p^2 + qV$$

$$\vec{p} = m\vec{v} = \vec{p}_{canonical} - q\vec{A}$$

$$\mathcal{H} = \frac{1}{2m}\left(\vec{p}_{canonical} - q\vec{A}\right)^2 + qV$$

now take $\vec{p}_{canonical} \Rightarrow -i\hbar\vec{\nabla}$

$$\mathcal{H}\psi = \frac{1}{2m}\left(-i\hbar\vec{\nabla} - q\vec{A}\right)^2\psi + qV\psi = E\psi$$

Schrodinger's equation where the term in $\vec{A}$ gives the effects on the eigenstates of the magnetic field $\vec{B}$, electromagnetic waves in a vacuum : for $\rho = \vec{j} = 0$

1) $\vec{\nabla}\cdot\vec{E} = 0$ 3) $\vec{\nabla}\cdot\vec{B} = 0$

2) $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ 4) $\vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$

$$\vec{\nabla}\times(2) \Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{E}}_{=0 \text{ as } \rho=0}) - \nabla^2\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$

from (4)

$$-\nabla^2\vec{E} = -\frac{\partial}{\partial t}\left(\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

$$\Rightarrow \left(\nabla^2 - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\right)\vec{E} = \Box^2\vec{E} = 0.$$

Similarly $\vec{\nabla}\times(4) \Rightarrow$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{B}}_{=0}) - \vec{\nabla}^2 B = \mu_0\varepsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

(2)

$$= \mu_0\varepsilon_0\frac{\partial}{\partial t}\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

$$\Rightarrow \left(\nabla^2 - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\right)\vec{B} = \Box^2\vec{B} = 0.$$

for any function $f(\vec{r},t)$,

$$\nabla^2 f - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$$ is "wave equation"
describes waves moving with speed v.

$\Rightarrow$ Maxwell's equ's in vacuum have wave solutions that move with speed $= \frac{1}{\sqrt{\varepsilon_0\mu_0}} = 3\times10^8$ m/s.

This combination of $\varepsilon_0\mu_0$ turns out to be exactly the speed of light! This realization of Maxwell's demonstrated that light was just an electro-magnetic wave!

Solutions to the wave equation $\Box^2 f(\vec{r},t) = 0$

plane waves if $g(\phi)$ is any function of a single variable ϕ

then define

$$f(\vec{r},t) = g(\vec{k}\cdot\vec{r} - \omega t)$$

where $\vec{k}$ is any constant vector and $\omega^2 = v^2 k^2$

Then $f(\vec{r},t)$ solves the wave equation!

proof: $\Box^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$

$$\frac{\partial f}{\partial x} = \frac{dg}{d\phi}\frac{\partial \phi}{\partial x} = \frac{dg}{d\phi} k_x \quad \text{by chain rule}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{dg}{d\phi} k_x\right) = \frac{d^2 g}{d\phi^2}\frac{\partial \phi}{\partial x} k_x = \frac{d^2 g}{d\phi^2} k_x^2$$

similarly $\frac{\partial^2 f}{\partial y^2} = \frac{d^2 g}{d\phi^2} k_y^2$, $\frac{\partial^2 f}{\partial z^2} = \frac{d^2 g}{d\phi^2} k_z^2$, $\frac{\partial^2 f}{\partial t^2} = \frac{d^2 g}{d\phi^2}\omega^2$

$$\text{so } \Box^2 f = \frac{d^2 g}{d\phi^2}\left(k_x^2 + k_y^2 + k_z^2 - \frac{\omega^2}{v^2}\right) = \frac{d^2 g}{d\phi^2}\left(k^2 - \frac{\omega^2}{v^2}\right) = 0$$

provided $\omega^2 = v^2 k^2$

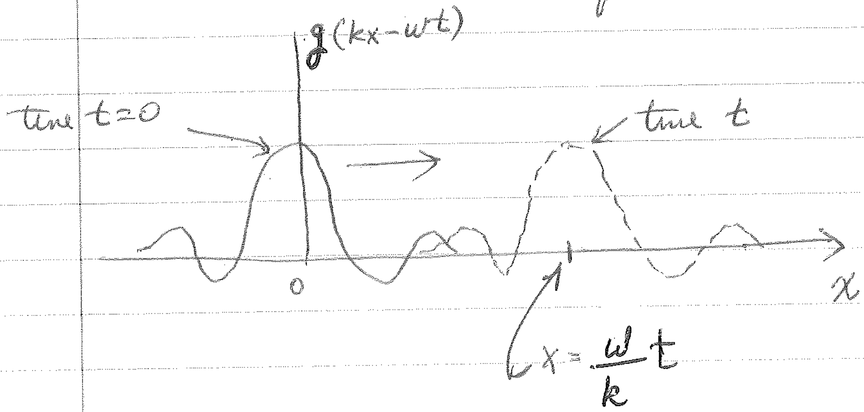
This type of solution is called a *plane wave* because $f(\vec{r},t)$ is constant on all planes $\perp$ to $\vec{k}$

ie if $\Delta\vec{r}$ is $\perp$ $\vec{k}$ $(\Delta\vec{r}\cdot\vec{k} = 0)$ then

$$f(\vec{r}+\Delta\vec{r},t) = g(\vec{k}\cdot\vec{r} + \underbrace{\vec{k}\cdot\Delta\vec{r}}_{=0} - \omega t) = g(\vec{k}\cdot\vec{r} - \omega t)$$

$$= f(\vec{r},t)$$

v is ~~velocit~~ speed of wave, suppose $\vec{k} = k\hat{x}$



$g(\phi)$ is called the <u>envelope</u> of the wave

curve shifted to right a distance $x = \frac{\omega}{k}t$ in time t.

$\Rightarrow$ curve moves with velocity $\vec{v} = \frac{\omega}{k}\hat{x}$

<u>spherical waves</u>

solution to wave eqn that depends only on radial coord r, ie $f(r,t)$ $\leftarrow$ f const on spheres of radius r

$$\left[\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right] f(r,t) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$$

try solution of form $\boxed{f(r,t) = \frac{g(kr-\omega t)}{r}}$

call $\phi = kr - \omega t$

$$r^2\left(\frac{\partial f}{\partial r}\right) = r^2\left(\frac{1}{r}\frac{dg}{d\phi}k - \frac{g}{r^2}\right) = r\frac{dg}{d\phi}k - g$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) = \frac{1}{r^2}\left[\frac{dg}{d\phi}k + r\frac{d^2g}{d\phi^2}k^2 - \frac{dg}{d\phi}k\right] = \frac{d^2g}{d\phi^2}\frac{k^2}{r}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{1}{r}\frac{d^2g}{d\phi^2}\omega^2$$

so $\frac{g}{r}(kr-\omega t)$ solves wave eqn provided $\frac{d^2g}{d\phi^2}\frac{k^2}{r} = \frac{1}{v^2}\frac{d^2g}{d\phi^2}\frac{\omega^2}{r}$

ie $\omega^2 = v^2k^2$

Note: if $f_1(\vec{r},t)$ and $f_2(\vec{r},t)$ are solutions to wave eqn, then so is $f_1 + f_2$, as $\Box^2$ is a <u>linear</u> operator

<u>Back to plane waves</u>

A particular solution : sinusoidal wave

$$f(\vec{r},t) = A\cos(\vec{k}\cdot\vec{r} - \omega t + \delta)$$

period of wave is $T = \frac{2\pi}{\omega}$: $f(\vec{r}, t+T) = f(t)$

frequency of wave is $\nu = \omega/2\pi = 1/T$

<u>angular freq</u> is ω

wavelength is $\lambda = \frac{2\pi}{|\vec{k}|}$: $f(\vec{r} + \lambda\hat{k}, t) = f(\vec{r}, t)$

<u>wave vector</u> is $\vec{k}$ <u>wave number</u> is $k = |\vec{k}|$

<u>amplitude</u> is A

<u>phase constant</u> is δ

phase velocity is $\vec{v} = \frac{\omega}{|\vec{k}|}\hat{k}$ travels in direction $\hat{k}$

<u>ex:</u> if $\vec{k} = k\hat{x}$ then wave travels in $+\hat{x}$ directi

if $\vec{k} = -k\hat{x}$ then $A\cos(-kx - \omega t + \delta)$

travels in $-\hat{x}$ direction

<u>Complex notation</u> $e^{i\theta} = \cos\theta + i\sin\theta$

$$f(\vec{r},t) = \mathrm{Re}\left[A e^{i(\vec{k}\cdot\vec{r} - \omega t + \delta)}\right]$$

usually we will not write the "Re" ~~by~~ but leave it implied.

$$f(\vec{r},t) = \mathrm{Re}\Big[\underbrace{A e^{i\delta}}_{\text{complex amplitude}} e^{i(\vec{k}\cdot\vec{r} - \omega t)}\Big]$$

Sinusoidal waves particularly important, since we can expand any solution of wave eqn in terms of them
$\Rightarrow$ theory of Fourier Transforms

Fourier Series: Any function $f(x)$ defined on $[-\frac{L}{2}, \frac{L}{2}]$ can be expressed as

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{2\pi n x}{L}\right) + B_n \sin\left(\frac{2\pi n x}{L}\right) \right]$$

where $$A_n = \frac{2}{L} \int_{-\frac{L}{2}}^{L/2} dx\, f(x) \cos\left(\frac{2\pi n x}{L}\right)$$

$$B_n = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x) \sin\left(\frac{2\pi n x}{L}\right)$$

rewrite in terms of complex exponential.

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left\{ \left(\frac{A_n}{2} + \frac{B_n}{2i} \right) e^{i\frac{2\pi n x}{L}} + \left(\frac{A_n}{2} - \frac{B_n}{2i} \right) e^{-i\frac{2\pi n x}{L}} \right\}$$

where
$$\frac{A_n}{2} \pm \frac{B_n}{2i} = \frac{A_n}{2} \mp \frac{iB_n}{2} = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)\, \frac{1}{2} \left\{ \cos\frac{2\pi n x}{L} \mp i \sin\frac{2\pi n x}{L} \right\}$$

$$= \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)\, e^{\mp i \frac{2\pi n x}{L}}$$

$n = 1, 2, \ldots$

Define $f_n \equiv \frac{A_n}{2} + \frac{B_n}{2i} = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)\, e^{-i\frac{2\pi n x}{L}}$

$$f_{-n} \equiv \frac{A_n}{2} - \frac{B_n}{2i} = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)\, e^{-i\frac{2\pi(-n)x}{L}}$$

$$f_0 \equiv \frac{A_0}{2} = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)$$

$$f(x) = f_0 + \sum_{n=1}^{\infty}\left\{ f_n e^{i\frac{2\pi n x}{L}} + f_{-n} e^{i 2\pi(-n) x/L} \right\}$$

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} f_n e^{i 2\pi n x/L}, \quad f_n = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)\, e^{-i\frac{2\pi n x}{L}}}$$

Fourier series in complex form

Now let $L \to \infty$

Define $k_n = \frac{2\pi n}{L}$, $k_{n+1} - k_n = \Delta k = \frac{2\pi}{L}$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{\Delta k}{2\pi} L f_n e^{i k_n x}$$

Define $\tilde{f}(k_n) \equiv \frac{L}{2\pi} f_n \equiv \frac{1}{2\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)\, e^{-i k_n x}$

$$f(x) = \sum_{n=-\infty}^{\infty} \Delta k\, \tilde{f}(k_n)\, e^{i k_n x}$$

as $L \to \infty$, $\Delta k \to 0$, $\sum \Delta k \to \int dk$

$$\boxed{f(x) = \int_{-\infty}^{\infty} dk\, \tilde{f}(k)\, e^{ikx}, \quad \tilde{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx}}$$

Fourier transform

$\tilde{f}(k)$ is the Fourier transform of $f(x)$.