$$f = f^*$$

If $f(x)$ is a <u>real</u> function, then

$$\tilde{f}(-k) = \int_{-\infty}^{\infty} \frac{dx}{2\pi} f(x) e^{+ikx} = \int_{-\infty}^{\infty} \frac{dx}{2\pi} f^*(x) \left(e^{-ikx}\right)^*$$

$$= \left(\int_{-\infty}^{\infty} \frac{dx}{2\pi} f(x) e^{-ikx}\right)^* = \tilde{f}^*(+k)$$

$$\boxed{\tilde{f}(-k) = \tilde{f}^*(k)}$$

For a function of 3-dim space,

$$f(x,y,z) = \int_{-\infty}^{\infty} dk_x\, dk_y\, dk_z\, \tilde{f}(k_x, k_y, k_z) e^{ik_x x} e^{ik_y y} e^{ik_z z}$$

or

$$\boxed{\begin{aligned} f(\vec{r}) &= \int_{-\infty}^{\infty} d^3k\, \tilde{f}(\vec{k}) e^{i\vec{k}\cdot\vec{r}} \\ \text{where}\quad \tilde{f}(\vec{k}) &= \int_{-\infty}^{\infty} \frac{d^3r}{(2\pi)^3} f(\vec{r}) e^{-i\vec{k}\cdot\vec{r}} \end{aligned}}$$

For function of $\vec{r}$ and t

$$f(\vec{r}, t) = \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega\, \tilde{f}(\vec{k}, \omega) e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

where

$$\tilde{f}(\vec{k}, \omega) = \int_{-\infty}^{\infty} \frac{d^3r}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dt}{2\pi} f(\vec{r}, t) e^{-i(\vec{k}\cdot\vec{r} - \omega t)}$$

Fourier transforms are of enormous help in solving partial differential equations.

General solution to wave equation

$\Box^2 f(\vec{r},t) = 0$ subst in F.T.

$$\Box^2 \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega\, \tilde{f}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega\, \tilde{f}(\vec{k},\omega) \left[\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right] e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\nabla^2 e^{i\vec{k}\cdot\vec{r}} = \vec{\nabla}\cdot\left(\vec{\nabla} e^{i\vec{k}\cdot\vec{r}}\right) = \vec{\nabla}\cdot\begin{pmatrix} \frac{\partial}{\partial x} e^{i(k_x x + k_y y + k_z z)} \\ \frac{\partial}{\partial y} e^{i(k_x x + k_y y + k_z z)} \\ \frac{\partial}{\partial z} e^{i(k_x x + k_y y + k_z z)} \end{pmatrix}$$

$$= \vec{\nabla}\cdot\begin{pmatrix} ik_x \\ ik_y \\ ik_z \end{pmatrix} e^{i\vec{k}\cdot\vec{r}} = \vec{\nabla}\cdot\left(i\vec{k}\, e^{i\vec{k}\cdot\vec{r}}\right) = i\vec{k}\cdot\vec{\nabla} e^{i\vec{k}\cdot\vec{r}}$$

product rule:

$$= i\vec{k}\cdot\vec{\nabla} e^{i\vec{k}\cdot\vec{r}} = (i\vec{k})\cdot(i\vec{k})\, e^{i\vec{k}\cdot\vec{r}} = -k^2 e^{i\vec{k}\cdot\vec{r}}$$

$$\frac{\partial^2}{\partial t^2} e^{-i\omega t} = -\omega^2 e^{-i\omega t}$$

$$\Rightarrow \left[\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right] e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \left(-k^2 + \frac{\omega^2}{v^2}\right) e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

So wave equ is

$$\int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega\, \tilde{f}(\vec{k},\omega)\left(-k^2 + \frac{\omega^2}{v^2}\right) e^{i(\vec{k}\cdot\vec{r}-\omega t)} = 0$$

$$\Rightarrow \tilde{f}(\vec{k},\omega)\left(-k^2 + \frac{\omega^2}{v^2}\right) = 0$$

$\because$ if a function $=0$, then its F.T. $=0$ (all Fourier coefficients $=0$)

either

$\Rightarrow \tilde{f}(\vec{k},\omega) = 0$, ie there is no such $(\vec{k},\omega)$ component

or $k^2 v^2 = \omega^2$. $\therefore$ for each $\vec{k}$, only $\omega = v|\vec{k}|$ is present

So most general solution is

$$f(\vec{r},t) = \int_{-\infty}^{\infty} d^3k \; \tilde{f}(\vec{k})\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \qquad \text{where } \omega^2 = v^2 k^2$$

[no longer integrate over ω since for each $\vec{k}$, ω fixed by]
and $\tilde{f}(\vec{k})$ is anything

General solution is ~~sum~~ linear combination of sinusoidal waves.

In most problems therefore, it will be enough to determine how plane sinusoidal waves with wavevector $\vec{k}$ behave. The general solution can then always be represented as a linear combination of these plain sinusoidal waves.

Note, if we know $\vec{f}(\vec{r}, t=0)$, and we know $f(\vec{r}, t)$ solves the wave equation, then we can use the above to find how f evolves in time, ie the initial condition determines the propagation of the wave.

Let $f(\vec{r}, t=0) = f_0(\vec{r})$

Define $\hat{f}_0(\vec{k}) = \int \frac{d^3r}{(2\pi)^3} f_0(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}$

and so $f_0(\vec{r}) = \int d^3k\, \hat{f}_0(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}$

Now $f(\vec{r}, 0) = \int d^3k\, \hat{f}_0(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}$

but under the wave equation each sinusoidal wave must propagate as $e^{i(\vec{k}\cdot\vec{r} - \omega t)}$ where $\omega = v|\vec{k}|$.

So we then must have

$$f(\vec{r}, t) = \int d^3k\, \tilde{f}_0(\vec{k})\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

and the initial condition $f_0(\vec{r}) = f(\vec{r}, t=0)$ determines the future evolution of the wave

Inhomogeneous wave equation

$$\Box^2 f(\vec{r},t) = g(\vec{r},t) \qquad \text{where } g \text{ is a given source function}$$

$$f(\vec{r},t) = \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \; \tilde{f}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$g(\vec{r},t) = \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \; \tilde{g}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

substitute in

$$\Box^2 f = \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \; \tilde{f}(\vec{k},\omega)\left[-k^2 + \frac{\omega^2}{v^2}\right] e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \; \tilde{g}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

equate Fourier components (if two functions are equal, then their Fourier transforms are equal)

$$\Rightarrow \tilde{f}(\vec{k},\omega)\left[-k^2 + \frac{\omega^2}{v^2}\right] = \tilde{g}(\vec{k},\omega)$$

$$\Rightarrow f(\vec{k},\omega) = \frac{\tilde{g}(\vec{k},\omega)}{\frac{\omega^2}{v^2} - k^2}$$

$$f(\vec{r},t) = \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \; \frac{\tilde{g}(\vec{k},\omega)}{\frac{\omega^2}{v^2} - k^2}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \; \frac{e^{i(\vec{k}\cdot\vec{r}-\omega t)}}{\frac{\omega^2}{v^2} - k^2} \cdot \underbrace{\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt' \; g(\vec{r}',t')\, e^{-i(\vec{k}\cdot\vec{r}'-\omega t')}}_{\tilde{g}(\vec{k},\omega)}$$

$$f(\vec{r},t) = \int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt' \, g(\vec{r}',t') \boxed{\int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \frac{1}{(2\pi)^4} \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} e^{-i\omega(t-t')}}{\left(\frac{\omega^2}{v^2} - k^2\right)}}$$

Green's function for wave equ

$G(\vec{r}-\vec{r}', t-t')$ – independent of the source $g(\vec{r},t)$

$$f(\vec{r},t) = \int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt' \; G(\vec{r}-\vec{r}', t-t') \, g(\vec{r}',t')$$

Same form as solution to Poissons Equ

$$-\nabla^2 V = \rho/\varepsilon_0 \Rightarrow V(\vec{r}) = \int d^3r \, \frac{\rho(\vec{r}')}{\varepsilon_0} \, G(\vec{r},\vec{r}')$$

$$G(\vec{r},\vec{r}') = \frac{1}{4\pi} \quad \frac{1}{|\vec{r}-\vec{r}'|}$$

Green's function for Poisson's equ

Polarization

vector wave $\vec{f}(\vec{r},t) = \vec{A}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

where $\vec{A} = A\hat{m}$, A may be complex number to include phase factor.

if $\hat{m}$ is $\parallel \hat{k}$ we have longitudinal polarization

" $\hat{m}$ is $\perp \hat{k}$ " " transverse polarization

$\cancel{\vec{\nabla}\cdot\vec{E} = 0} \quad \cancel{\vec{\nabla}\cdot\vec{B} = 0}$

$\cancel{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}} \quad \cancel{\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t}}$

Circular polarization: sum of orthogonal transverse polarizations, $\pi/2$ out of phase

Consider $\vec{f}(\vec{r},t) = A\hat{m}_1 e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \underbrace{iA\hat{m}_2 e^{i(\vec{k}\cdot\vec{r}-\omega t)}}_{A\hat{m}_2 e^{i(\vec{k}\cdot\vec{r}-\omega t+\frac{\pi}{2})}}$

where $\hat{m}_1 \perp \hat{m}_2 \perp \hat{k}$ is right handed coord system

$$\vec{f} = A(\hat{m}_1 + i\hat{m}_2)e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

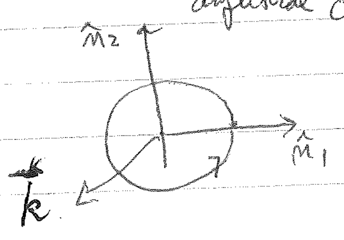
to get physical field, take Real part of complex form

$$\Rightarrow \vec{f} = A\hat{m}_1 \cos(\vec{k}\cdot\vec{r}-\omega t) - A\hat{m}_2 \sin(\vec{k}\cdot\vec{r}-\omega t)$$

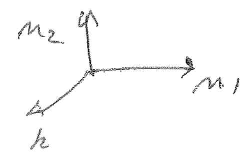
$$= A\hat{m}_1 \cos(\omega t - \vec{k}\cdot\vec{r}) + A\hat{m}_2 \sin(\omega t - \vec{k}\cdot\vec{r})$$

$$|\vec{f}|^2 = A^2\cos^2(\vec{k}\cdot\vec{r}-\omega t) + A^2\sin^2(\vec{k}\cdot\vec{r}-\omega t) = A^2$$

amplitude constant, but direction of $\vec{f}$ rotates in time counterclockwise with frequency ω.



$\vec{f} = A(\hat{n}_1 + i\,\hat{n}_2)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$, where $\hat{n}_1, \hat{n}_2, \hat{k}$ form right handed coord system



is a "Right handed" circularly polarized wave

$\vec{f} = A(\hat{n}_1 - i\,\hat{n}_2)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ is a "Left handed" circularly polarized wave – direction of $\vec{f}$ rotates in time clockwise

Plane EM waves in a vacuum

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

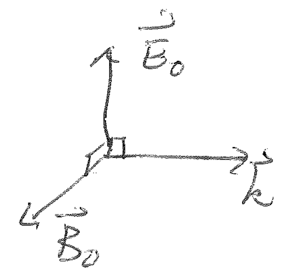
Assume solutions of form $\left.\begin{aligned}\vec{E} &= \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \\ \vec{B} &= \vec{B}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\end{aligned}\right\}$ where $\omega = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} k = ck$

Maxwell's equ's becom

1) $i\vec{k}\cdot\vec{E}_0 = 0$ 3) $i\vec{k}\cdot\vec{B}_0 = 0$

2) $i\vec{k}\times\vec{E}_0 = +i\omega\vec{B}_0$ 4) $i\vec{k}\times\vec{B}_0 = \mu_0\varepsilon_0(-i\omega)\vec{E}_0$

(1) and (3) $\Rightarrow$ EM waves are transverse polarized. $\vec{E}_0$ and $\vec{B}_0$ both $\perp$ to $\vec{k}$.



2) $\Rightarrow$ $\vec{B}_0 = \frac{\vec{k}}{\omega}\times\vec{E}_0 = \frac{1}{c}\hat{k}\times\vec{E}_0 \Rightarrow \vec{B}_0 \perp \vec{E}_0$

$|\vec{B}_0| = \frac{1}{c}|\vec{E}_0|$

$\uparrow$ very important factor $\frac{1}{c}$!

Since Lorentz force is $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$, the force on a charged particle due to an electromagnetic wave is predominantly from the electric field $\vec{E}$. The force due to the magnetic field is $\sim vB_0 = \left(\frac{v}{c}\right)E_0$, is reduced by a factor $\left(\frac{v}{c}\right) \ll 1$, unless charge is moving relativistically fast.