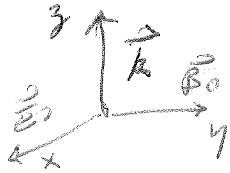


$\vec{k} = k\hat{z}$

energy + momentum in EM wave:

$\vec{E}(r,t) = E_0 \cos(kz - \omega t)\,\hat{x}$ $\qquad$ $\vec{B}(r,t) = \frac{1}{c} E_0 \cos(kz - \omega t)\,\hat{y}$

energy density: Remember, if using complex exponential forms one must take the real part before computing products!

$$u_{EB} = \frac{\varepsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 = \frac{\varepsilon_0}{2} E_0^2 \cos^2(kz - \omega t) + \frac{1}{2\mu_0 c^2} E_0^2 \cos^2(kz - \omega t)$$

$$= \frac{1}{2} E_0^2 \cos^2(kz - \omega t) \underbrace{\left[\varepsilon_0 + \frac{1}{\mu_0 c^2} \right]}_{\underbrace{\varepsilon_0 + \frac{\mu_0 \varepsilon_0}{\mu_0}}_{2\varepsilon_0}} \qquad \text{use } c^2 = \frac{1}{\mu_0 \varepsilon_0}$$

$$u_{EB} = \varepsilon_0 E_0^2 \cos^2(kz - \omega t)$$

* Note: when taking the product of 2 factors of $\vec{E}$ or $\vec{B}$, important to take Re parts first, if using complex notation

energy current

$$\vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B})$$

$$= \frac{1}{\mu_0 c} E_0^2 \cos^2(kz - \omega t) (\hat{x} \times \hat{y}) = c\,\varepsilon_0 E_0^2 \cos^2(kz - \omega t)\,\hat{z}$$

using $\frac{1}{\mu_0 c} = \frac{c}{\mu_0 c^2} = \frac{c\,\mu_0 \varepsilon_0}{\mu_0} = c\,\varepsilon_0$

$$\vec{S} = c\, u_{EB}\, \hat{z}$$

momentum density $\quad \vec{p}_{EB} = \frac{1}{c^2} \vec{S} = \frac{u_{EB}}{c}\, \hat{z}$

$\Rightarrow \quad u_{EB} = c\,|\vec{p}_{EB}|$ $\quad$ – energy-momentum relation of photons

Since for visible light, $\lambda \sim 5 \times 10^{-7}$ m ~ 5000 Å

$$T = \frac{\lambda}{c} = \frac{5 \times 10^{-7}}{3 \times 10^{8}} \text{ sec} = 1.6 \times 10^{-15} \text{ sec}$$

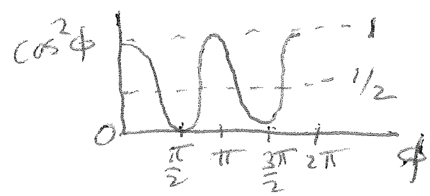
for most classical measurements, on macroscopic scale, the measurement will average over many oscillations of the wave. Therefore one is interested in averages

$$\langle u_{EB}\rangle = \frac{1}{T}\int_0^T dt\; u_{EB} \qquad \text{average over one period of oscillation}$$

$$= \frac{\lambda}{c}\int_0^T dt\; \varepsilon_0 E_0^2 \cos^2(kz-\omega t) \qquad \boxed{T = \frac{2\pi}{\omega} \text{ is period of oscillation} \;= \frac{\lambda}{c}}$$

$$\langle u_{EB}\rangle = \frac{1}{2}\varepsilon_0 E_0^2$$

average of $\cos^2(\phi)$ over one period is $\frac{1}{2}$



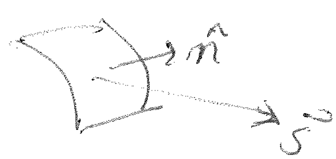
$$\langle \vec{S}\rangle = c\langle u_{EB}\rangle \hat{z}$$

$$\langle \vec{p}_{EB}\rangle = \frac{1}{c}\langle u_{EB}\rangle \hat{z}$$

<u>intensity</u> = average (over time) power per area transported by wave

intensity $\quad I = |\langle \vec{S}\rangle|$ magnitude of energy current
~ (amplitude of field)^2

$\langle \vec{S}\rangle \cdot \hat{n}$ = average power per area transported through surface with normal $\hat{n}$



Maxwell's Eqn in Matter

$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho_{tot}$ $\qquad$ $\vec{\nabla} \cdot \vec{B} = 0$

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ $\qquad$ $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_{tot} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

want to write $\rho_{tot} = \rho_{free} + \underbrace{\rho_b}_{\text{bound charge}}$ $\qquad$ $\vec{J}_{tot} = \vec{J}_{free} + \underbrace{\vec{J}_b}_{\text{bound current}}$

in statics: $\rho_b = -\vec{\nabla} \cdot \vec{P}$ $\qquad$ $\vec{J}_b = \vec{\nabla} \times \vec{M}$

in dynamics: conservation of bound charge $\Rightarrow \vec{\nabla} \cdot \vec{J}_b = -\frac{\partial \rho_b}{\partial t}$

$$\underbrace{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{M})}_{= 0} = + \underbrace{\frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{P})}_{\neq 0}$$

something must be missing! The bound current arising from $\vec{M}$ must not be all the bound current. There must be bound current arising from a time varying $\vec{P}$.

bound current from polarization, $\vec{J}_p$ must satisfy

$$\vec{\nabla} \cdot \vec{J}_p = -\frac{\partial \rho_b}{\partial t} = \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{P} = \nabla \cdot \left(\frac{\partial \vec{P}}{\partial t}\right)$$

$$\Rightarrow \vec{J}_p = \frac{\partial \vec{P}}{\partial t}$$

$$\Rightarrow \vec{J}_b = \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t}$$

$$\vec\nabla\cdot\vec E = \frac{1}{\varepsilon_0}\left(\rho_f - \vec\nabla\cdot\vec P\right) \qquad \vec\nabla\cdot\vec B = 0$$

$$\vec\nabla\times\vec B = \mu_0\left(\vec J_f + \vec\nabla\times\vec M + \frac{\partial\vec P}{\partial t}\right) + \mu_0\varepsilon_0\frac{\partial\vec E}{\partial t} \qquad \vec\nabla\times\vec E = -\frac{\partial\vec B}{\partial t}$$

define $\vec D = \varepsilon_0\vec E + \vec P$

$\vec H = \frac{1}{\mu_0}\vec B - \vec M$

$$\Rightarrow \quad \vec\nabla\cdot\vec D = \rho_f \qquad \vec\nabla\times\vec H = \vec J_f + \frac{\partial\vec D}{\partial t}$$

inhomogeneous equ[ations] involve the sources

$$\vec\nabla\cdot\vec B = 0 \qquad \vec\nabla\times\vec E = -\frac{\partial\vec B}{\partial t}$$

homogeneous equ[ations]

for linear materials, $\left.\begin{array}{l}\vec D = \varepsilon\vec E \\ \vec H = \frac{1}{\mu}\vec B\end{array}\right\}$ closes above equations.

$$\vec\nabla\cdot\vec E = \frac{1}{\varepsilon_0}\left(\rho_f - \vec\nabla\cdot\vec P\right) \Rightarrow \vec\nabla\cdot\left(\varepsilon_0\vec E + \vec P\right) = \rho_f$$

$$\Rightarrow \vec\nabla\cdot\vec D = \rho_f$$

$$\vec\nabla\times\vec B = \mu_0\left(\vec J_f + \vec\nabla\times\vec M + \frac{\partial\vec P}{\partial t}\right) + \mu_0\varepsilon_0\frac{\partial\vec E}{\partial t}$$

$$\Rightarrow \vec\nabla\times\left(\frac{\vec B}{\mu_0} - \vec M\right) = \vec J_f + \frac{\partial}{\partial t}\left(\varepsilon_0\vec E + \vec P\right)$$

$$\Rightarrow \vec\nabla\times\vec H = \vec J_f + \frac{\partial\vec D}{\partial t}$$

If we had $\vec{D}(\vec{r},t) = \varepsilon E(\vec{r},t)$
$\vec{H}(\vec{r},t) = \frac{1}{\mu} B(\vec{r},t)$

then Maxwell's eqns, in absense of free charge + free current would be

$$\varepsilon \vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu\varepsilon \frac{\partial \vec{E}}{\partial t}$$
$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

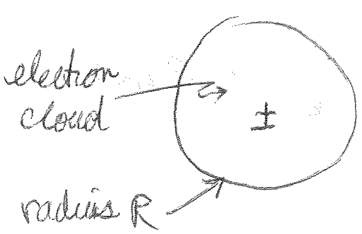
everything would be the same except $\varepsilon_0\mu_0 \to \varepsilon\mu > \varepsilon_0\mu_0$
the speed of EM waves in the material would be

$$v = \frac{1}{\sqrt{\varepsilon\mu}} < c \qquad c/v \equiv n \quad \text{index of refraction}$$

would have $|\vec{B}| = \frac{1}{v}|\vec{E}|$

In general however, things are <u>much more complicated</u> for time varying response

Consider model for polarization of a neutral atom, that we saw last semester



If displace center of electron cloud from ion by distance $\vec{r}$, then there is a restoring force

$$\vec{F}_{rest} = -\frac{e^2 \vec{r}}{4\pi\varepsilon_0 R^3} \equiv -m\omega_0^2 \vec{r}$$

imagine electron is a cloud of uniform charge density total charge q

(electric field from electron cloud increases linearly with distance from origin)

electron mass

ω_0 has units of frequency.

For a static $\vec{E}$ field we then get

force balance $\Rightarrow$ $-e\vec{E} + \vec{F}_{rest} = 0$

$$\Rightarrow -e\vec{E} = m\omega_0^2 \vec{r}$$

$$\Rightarrow \vec{r} = \frac{-e}{m\omega_0^2} \vec{E}$$

dipole moment $\vec{p} = -e\vec{r} = \frac{e^2}{m\omega_0^2} \vec{E} = \alpha \vec{E}$

atomic polarizeability $\alpha = \frac{e^2}{m\omega_0^2}$

polarization density $\vec{P} \simeq N\alpha$ $\qquad N$ = atomic density

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + N\alpha \vec{E}$$

$$= \varepsilon_0 \left(1 + \frac{N\alpha}{\varepsilon_0}\right) \vec{E}$$

$$= \varepsilon \vec{E}$$

$$\varepsilon = \varepsilon_0 \left(1 + \frac{N\alpha}{\varepsilon_0}\right)$$

$\uparrow$ χ_e electric susceptibility

$K = 1 + \chi_e = 1 + \frac{N\alpha}{\varepsilon_0}$ dielectric constant

Looks like ε is a constant!