But in a dynamic situation, with time varying $\vec{E}(t)$ there in general will also be a damping, or friction force, due to energy transfer from atom to other degrees of freedom

$$\vec{F}_{damp} = -m\gamma \frac{d\vec{r}}{dt} \qquad \text{friction} \sim \text{velocity}$$

If electron is in external electric field, the eqn of motion is then

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}_{tot} = -e\vec{E}(t) - m\omega_0^2\vec{r} - m\gamma\frac{d\vec{r}}{dt}$$

or $\ddot{\vec{r}} + \gamma\dot{\vec{r}} + \omega_0^2\vec{r} = -\frac{e\vec{E}(t)}{m}$ (assuming that $\vec{E}$ is constant over spatial distances that electron moves)

driven damped harmonic oscillator!

Consider sinusoidal $\vec{E}$ field, in complex form.

$$\vec{E}(t) = \vec{E}_\omega e^{-i\omega t}$$

Assume solution $\vec{r}(t) = \vec{r}_\omega e^{-i\omega t}$

$$-\omega^2\vec{r}_\omega - i\omega\gamma\vec{r}_\omega + \omega_0^2\vec{r}_\omega = -\frac{e\vec{E}_\omega}{m}$$

$$\vec{r}_\omega = \frac{-e}{m(\omega_0^2 - \omega^2 - i\omega\gamma)}\vec{E}_\omega$$

dipole moment $\vec{p}(t) = -e\vec{r}(t) = \vec{p}_\omega e^{-i\omega t}$

$$\vec{p}_\omega = \frac{e^2}{m}\frac{1}{(\omega_0^2 - \omega^2 - i\omega\gamma)}\vec{E}_\omega$$

↑ resonance at $\omega \approx \omega_0$
width of resonance is γ

frequency dependent polarizability

$$\vec{p}_\omega = \alpha(\omega)\vec{E}_\omega \qquad \alpha(\omega) = \frac{e^2}{m}\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

varies with ω

Since $\alpha(\omega)$ is complex the polarization does not in general oscillate in phase with the electric field.

For a pure harmonic $\vec{E}$, $\vec{p}(t) = \alpha(\omega)\vec{E}_\omega e^{-i\omega t}$

$$= |\alpha(\omega)|\vec{E}_\omega e^{-i(\omega t - \delta)}$$

phase shift from $\vec{E}$

where δ is the phase of complex α

ie $\alpha = |\alpha| e^{i\delta}$

For a general electric field $\vec{E}(t) = \int_{-\infty}^{\infty} d\omega\, \vec{E}_\omega e^{-i\omega t}$

the response is $\vec{p}(t) = \int_{-\infty}^{\infty} d\omega\, \vec{p}_\omega e^{-i\omega t} = \int_{-\infty}^{\infty} d\omega\, \alpha(\omega)\vec{E}_\omega e^{-i\omega t}$

substitute in $\vec{E}_\omega = \int_{-\infty}^{\infty} \frac{dt'}{2\pi} \vec{E}(t') e^{i\omega t'}$ to get

$$\vec{p}(t) = \int_{-\infty}^{\infty} \frac{dt'}{2\pi} \vec{E}(t') \int_{-\infty}^{\infty} d\omega\, \alpha(\omega) e^{-i\omega(t-t')}$$

define Fourier transform $\tilde{\alpha}(t) \equiv \int_{-\infty}^{\infty} d\omega\, \alpha(\omega) e^{-i\omega t}$

$$\vec{p}(t) = \int_{-\infty}^{\infty} \frac{dt'}{2\pi} \vec{E}(t')\tilde{\alpha}(t-t')$$

$\vec{p}$ at t is due to $\vec{E}$ at all other times t', not only at time t

---

True in general: if $\tilde{A}(\omega) = \tilde{a}(\omega)\tilde{B}(\omega)$ is relation between Fourier transforms, then in time,

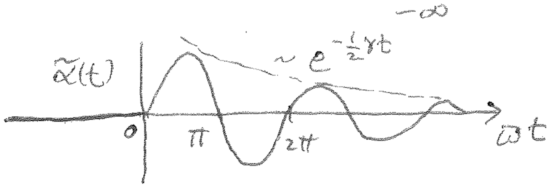
$$A(t) = \int_{-\infty}^{\infty} \frac{dt'}{2\pi} B(t') a(t-t')$$

$$\vec{p}(t) = \int_{-\infty}^{\infty} \frac{dt'}{2\pi}\, \vec{E}(t')\, \tilde{\alpha}(t-t')$$ response is <u>non-local</u> in time

i.e. $\vec{p}(t)$ is determined not just by the instaneous $\vec{E}(t)$, but by $\vec{E}(t')$ at other times $t' \neq t$.

For our simple model, $\alpha(\omega) = \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$

$$\tilde{\alpha}(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, \alpha(\omega) = \begin{cases} 2\pi \frac{e^2}{m} \frac{1}{\bar{\omega}} e^{-\frac{1}{2}\gamma t} \sin(\bar{\omega} t) & t>0 \\ 0 & t<0 \end{cases}$$



where $\bar{\omega} = \sqrt{\omega_0^2 - \frac{1}{4}\gamma^2}$

$\tilde{\alpha}(t) = 0$ for $t<0$ $\Rightarrow$ causal response, i.e. $\vec{p}(t)$ depends on $\vec{E}(t')$ only for <u>earlier</u> times $t' < t$

$\tilde{\alpha}(t)$ gives the polarization that results from a δ-function pulse in $\vec{E}$ at time $t'=0$, i.e. $\vec{E}(t') = \vec{E}_0\, \delta(t')$

$\tilde{\alpha}(t)$ has the familiar form of the displacement of a damped harmonic oscillator that is given an impulse kick at $t'=0$

For a dielectric, we now expect <u>polarization density</u> from a pure sinusoidal electric field, will be

$\vec{\mathbb{P}}(t) = \vec{\mathbb{P}}_\omega e^{-i\omega t}$ with $\vec{\mathbb{P}}_\omega = \varepsilon_0 \chi_e(\omega) \vec{E}_\omega$

$\chi_e(\omega)$ freq dependent electric susceptibility

where $\chi_e(\omega) \approx \frac{N\alpha(\omega)}{\varepsilon_0}$ for a dilute density of atoms

N = atomic density
i.e. atoms per volume

$\Rightarrow$ Displacement $\vec{D}(t) = \vec{D}_\omega e^{-i\omega t}$ with $\vec{D}_\omega = \varepsilon_0 \vec{E}_\omega + \vec{P}_\omega$

$$= \varepsilon_0 (1 + \chi_e(\omega)) \vec{E}_\omega$$

$$= \varepsilon(\omega) \vec{E}_\omega$$

$$\varepsilon(\omega) = \varepsilon_0 (1 + \chi_e(\omega))$$

$$= \varepsilon_0 K(\omega)$$

complex, freq dependent permeativity

freq dependent electric susceptibility $\chi_e(\omega) \approx \frac{N\alpha(\omega)}{\varepsilon_0}$ ← freq dependent atomic polarizability

freq dependent dielectric function

$$\vec{D}_\omega = \varepsilon(\omega) \vec{E}_\omega \Rightarrow \vec{D}(t) \neq \varepsilon \vec{E}(t) \text{, but rather}$$

$$\vec{D}(t) = \int_{-\infty}^{\infty} \frac{dt'}{2\pi} \vec{E}(t') \tilde{\varepsilon}(t-t')$$

← F.T. of $\varepsilon(\omega)$

ε complex $\Rightarrow \vec{D}_\omega$ and $\vec{E}_\omega$ are not in general in phase with each other

$\vec{D}(t)$ and $\vec{E}(t)$ are <u>non-locally</u> related in time

$\Rightarrow$ Maxwell's equations look very complicated when expressed in terms of time. For example:

assume $\mu = \mu_0$, $\vec{J}_{free} = 0$, then Ampere's Law is

$$\cancel{\frac{1}{\mu_0} \vec{\nabla} \times \vec{B} = \frac{\partial \vec{D}}{\partial t} = \int \frac{dt'}{2\pi} \vec{E}(t')}$$

$$\frac{1}{\mu_0} \vec{\nabla} \times \vec{B}(\vec{r},t) = \frac{\partial \vec{D}(\vec{r},t)}{\partial t} = \int_{-\infty}^{\infty} \frac{dt'}{2\pi} \vec{E}(\vec{r},t') \frac{d}{dt} \tilde{\varepsilon}(t-t')$$

becomes an integro-differential equation when expressed in terms of $\vec{B}$ and $\vec{E}$.

(Alternatively, Faraday's Law $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, would become an integro-differential equation if expressed in terms of $\vec{H}$ and $\vec{D}$)

Do the fields $\vec{E}$ and $\vec{B}$ inside such a material with an $\varepsilon(\omega)$ obey the wave equation when $\rho_f = 0$, $\vec{J}_f = 0$?

For simplicity we take a material with no magnetic response
$$\mu = \mu_0 \quad , \quad \vec{H} = \frac{\vec{B}}{\mu_0}$$

Ampere's law when $\vec{J}_f = 0$ is

$$\vec{\nabla}\times\vec{H} = \frac{\partial \vec{D}}{\partial t} \quad \Rightarrow \quad \vec{\nabla}\times\vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t}$$

use $\vec{D}(\vec{r},t) = \int_{-\infty}^{\infty} \frac{dt'}{2\pi}\, E(r,t')\, \tilde{\varepsilon}(t-t')$ where $\tilde{\varepsilon}(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, \varepsilon(\omega)$ is F.T. of $\varepsilon(\omega)$

Then
$$\vec{\nabla}\times\vec{B} = \mu_0 \frac{\partial}{\partial t} \int \frac{dt'}{2\pi}\, \vec{E}(r,t')\, \tilde{\varepsilon}(t-t') = \mu_0 \int \frac{dt'}{2\pi}\, \vec{E}(\vec{r},t')\, \frac{\partial \tilde{\varepsilon}(t-t')}{\partial t}$$

Take curl of both sides
$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = -\nabla^2 \vec{B} = \mu_0 \int \frac{dt'}{2\pi} \left[\vec{\nabla}\times\vec{E}(\vec{r},t')\right] \frac{\partial \tilde{\varepsilon}(t-t')}{\partial t}$$

use $\vec{\nabla}\times\vec{E}(\vec{r},t') = -\frac{\partial \vec{B}(r,t')}{\partial t'}$ Faraday's law

$$-\nabla^2 \vec{B}(\vec{r},t) = -\mu_0 \int \frac{dt'}{2\pi}\, \frac{\partial \vec{B}(\vec{r},t')}{\partial t'}\, \frac{\partial \tilde{\varepsilon}(t-t')}{\partial t}$$

use $\frac{\partial \tilde{\varepsilon}(t-t')}{\partial t} = -\frac{\partial \tilde{\varepsilon}(t-t')}{\partial t'}$

$$-\nabla^2 \vec{B}(\vec{r},t) = \mu_0 \int_{-\infty}^{\infty} \frac{dt'}{2\pi}\, \frac{\partial \vec{B}(\vec{r},t')}{\partial t'}\, \frac{\partial \tilde{\varepsilon}(t-t')}{\partial t'}$$

integrate by parts
$$-\nabla^2 \vec{B}(\vec{r},t) = -\mu_0 \int \frac{dt'}{2\pi}\, \frac{\partial^2 \vec{B}(\vec{r},t')}{\partial t'^2}\, \tilde{\varepsilon}(t-t')$$

$$\nabla^2 \vec{B}(\vec{r},t) - \mu_0 \int \frac{dt'}{2\pi} \frac{\partial^2 B(\vec{r},t')}{\partial t'^2} \tilde{\varepsilon}(t-t') = 0$$

looks some what like the wave equation, except for the integration.

Suppose $\tilde{\varepsilon}(t-t') = 2\pi\varepsilon\,\delta(t-t')$ with ε a constant

Then we would get

$$\nabla^2 \vec{B}(\vec{r},t) - \mu_0 \varepsilon \int_{-\infty}^{\infty} dt' \frac{\partial^2 \vec{B}(\vec{r},t')}{\partial t'^2} \delta(t-t') = 0$$

$$\Rightarrow \nabla^2 B(\vec{r},t) - \mu_0 \varepsilon \frac{\partial^2 \vec{B}(\vec{r},t)}{\partial t^2} = 0 \qquad \text{The wave equation!}$$

So only if the responce function $\tilde{\varepsilon}(t) = 2\pi\varepsilon\,\delta(t)$, ie it is instantaneous in time, will $\vec{B}$ solve the wave equation with wave velocity $v = \frac{1}{\sqrt{\mu_0 \varepsilon}}$

What $\varepsilon(\omega)$ corresponds to $\tilde{\varepsilon}(t-t') = 2\pi\varepsilon\,\delta(t-t')$?

$$\varepsilon(\omega) \equiv \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{i\omega t} \tilde{\varepsilon}(t) \qquad \text{definition of Fourier transform}$$

$$= \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{-i\omega t} 2\pi\varepsilon\,\delta(t) = \varepsilon$$

So only when $\varepsilon(\omega) = \varepsilon$ is a constant indep of ω, will the fields obey the wave equation!

$\Longrightarrow$ Maxwell's Eqn only look simple when expressed in terms of <u>Fourier Transforms</u>.

For pure sinusoidal solutions: $\vec{E}(\vec{r},t) = \vec{E}_\omega\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$\vec{B}(\vec{r},t) = \vec{B}_\omega\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$\vec{H}(\vec{r},t) = \vec{H}_\omega\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$\vec{D}(\vec{r},t) = \vec{D}_\omega\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

for EM waves in dielectric, assume $\rho_f = \vec{J}_f = 0$

Maxwell's Eqns: $\vec{\nabla}\cdot\vec{D} = 0$, $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$, $\vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t}$

assume $\mu = \mu_0 \Rightarrow \vec{H}_\omega = \frac{1}{\mu_0}\vec{B}_\omega$

dielectric response given by $\varepsilon(\omega) \Rightarrow \vec{D}_\omega = \varepsilon(\omega)\vec{E}_\omega$

<u>For $\rho_f = \vec{J}_f = 0$</u>, Maxwell's Eqns in terms of the Fourier amplitudes are then

1) $i\vec{k}\cdot\vec{D}_\omega = i\varepsilon(\omega)\,\vec{k}\cdot\vec{E}_\omega = 0 \quad \Rightarrow \vec{k}\cdot\vec{E}_\omega = 0 \quad \vec{k}\perp\vec{E}_\omega$

2) $i\vec{k}\cdot\vec{B}_\omega = 0 \quad \Rightarrow \vec{k}\cdot\vec{B}_\omega = 0 \quad \vec{k}\perp\vec{B}_\omega$

transverse

3) Faraday $\quad i\vec{k}\times\vec{E}_\omega = i\omega\vec{B}_\omega$

4) Ampere $\quad i\vec{k}\times\vec{H}_\omega = -i\omega\vec{D}_\omega \Rightarrow \frac{i\vec{k}}{\mu_0}\times\vec{B}_\omega = -i\omega\varepsilon(\omega)\vec{E}_\omega$

$\vec{k}\times(\text{Faraday}) = i\vec{k}\times(\vec{k}\times\vec{E}_\omega) = i\omega(\vec{k}\times\vec{B}_\omega)$ substitute in from Ampere

$= -i\omega^2\varepsilon(\omega)\mu_0\vec{E}_\omega$

$$\vec{k}\times(\vec{k}\times\vec{E}_\omega) = \vec{k}\,\underbrace{(\vec{k}\cdot\vec{E}_\omega)}_{=0 \text{ by } (1)} - \vec{E}_\omega(\vec{k}\cdot\vec{k}) = -\omega^2\varepsilon(\omega)\mu_0\vec{E}_\omega$$

$$\Rightarrow k^2\vec{E}_\omega = \omega^2\varepsilon(\omega)\mu_0\vec{E}_\omega$$

$$\Rightarrow \boxed{\begin{aligned} k^2 &= \omega^2 \varepsilon(\omega)\mu_0 \\ k^2 &= \frac{\omega^2}{c^2}\left(\frac{\varepsilon(\omega)}{\varepsilon_0}\right) \end{aligned}}$$

use $\frac{1}{c^2} = \mu_0 \varepsilon_0$

"dispersion" relation for waves in dielectric

dispersion relation determines wave vector k, for a given frequency ω.

Note $\frac{\omega^2}{k^2} \neq$ constant $\Rightarrow \vec{E}$ is <u>not</u> solution of a wave equation $\Box^2 \vec{E} = 0$.

different frequencies travel with <u>different</u> speeds.

Since $\varepsilon(\omega)$ is complex $\quad \varepsilon(\omega) = \underbrace{\varepsilon_1(\omega)}_{Re[\varepsilon]} + i \underbrace{\varepsilon_2(\omega)}_{Im[\varepsilon]}$

then in general the wavevector is also complex

$$k = k_1 + i k_2 = \pm \frac{\omega}{c} \sqrt{\frac{\varepsilon_1}{\varepsilon_0} + i \frac{\varepsilon_2}{\varepsilon_0}}$$

For a wave traveling in $\hat{z}$ direction, $\vec{k} = k\hat{z}$, we have

$$\vec{E}(\vec{r},t) = \vec{E}_\omega e^{i(\vec{k}\cdot\vec{r} - \omega t)} = \vec{E}_\omega e^{i((k_1 + i k_2)z - \omega t)}$$

$$= \vec{E}_\omega e^{-k_2 z} e^{i(k_1 z - \omega t)}$$

If choose $+\sqrt{\ }$ solution for k_1, so that wave propagates in $+\hat{z}$ direction, then should take $+\sqrt{\ }$ solution for k_2, so that wave decays as it propagates into material