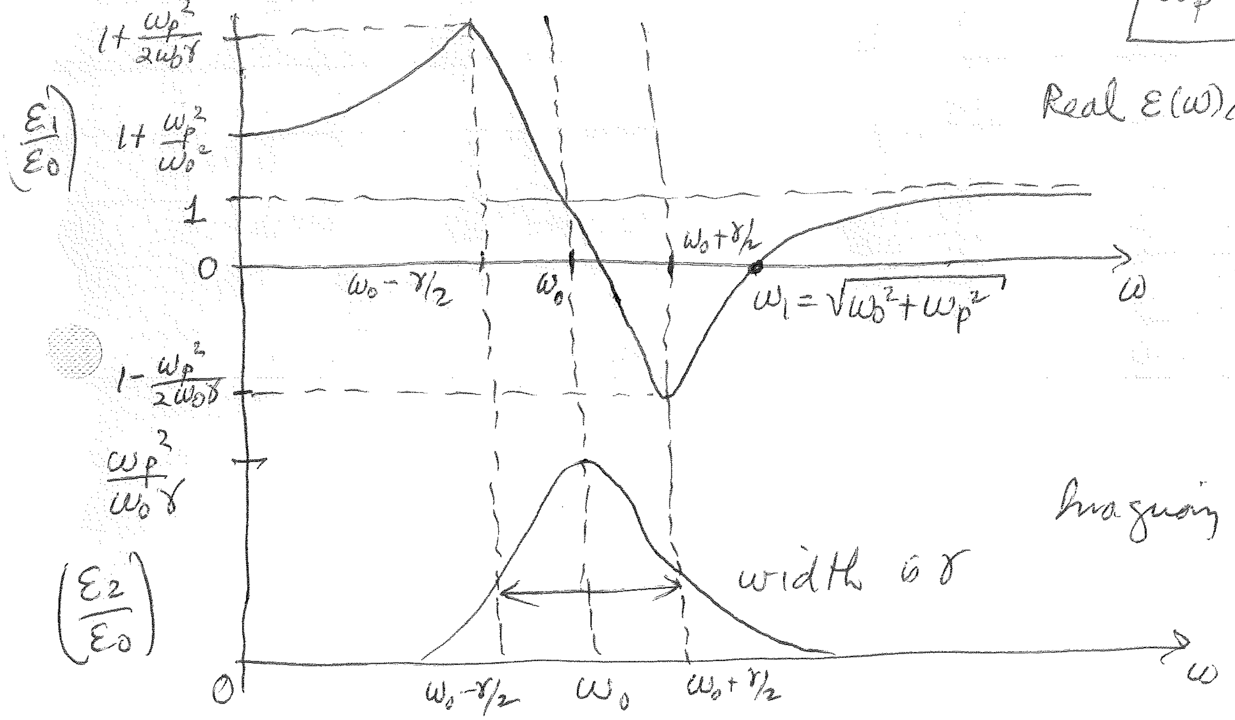
For our simple model

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{Ne^2}{m\varepsilon_0}\frac{1}{\omega_0^2-\omega^2-i\omega\gamma}$$

$$\Rightarrow \quad \frac{\varepsilon_1}{\varepsilon_0} = 1 + \frac{Ne^2}{m\varepsilon_0}\frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}$$ real part ε

$$\frac{\varepsilon_2}{\varepsilon_0} = \frac{Ne^2}{m\varepsilon_0}\frac{\omega\gamma}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}$$ Imaginary part ε

$\boxed{\omega_p^2 \equiv \frac{Ne^2}{m\varepsilon_0}}$ plasma freq



as $\left(\frac{\gamma}{\omega_0}\right)\to 0$, width of resonance decreases
height of peaks diverges

Notes for sketch

max and min of $\varepsilon_1/\varepsilon_0$ occur when $\dfrac{\partial(\varepsilon_1/\varepsilon_0)}{\partial\omega} = 0$

$$\Rightarrow \left[(\omega_0^2-\omega^2)^2+\omega^2\gamma^2\right](-2\omega) - (\omega_0^2-\omega^2)\left[2(\omega_0^2-\omega^2)(-2\omega)+2\omega\gamma^2\right]=0$$

$$(\omega_0^2-\omega^2)^2+\omega^2\gamma^2 - 2(\omega_0^2-\omega^2)^2 + (\omega_0^2-\omega^2)\gamma^2 = 0$$

$$(\omega_0^2-\omega^2)^2 = \omega_0^2\gamma^2$$

$$|\omega_0^2-\omega^2| = \omega_0\gamma$$

$$|\omega_0-\omega|(\omega_0+\omega) = \omega_0\gamma$$

for sharp resonance, peaks are when $\dfrac{\omega-\omega_0}{\omega_0} \ll 1 \rightarrow \omega_0+\omega \approx 2\omega_0$

$$\Rightarrow |\omega_0-\omega|\, 2\omega_0 = \omega_0\gamma$$

$$|\omega_0-\omega| = \frac{\gamma}{2} \Rightarrow \boxed{\omega-\omega_0 = \pm\frac{\gamma}{2}}$$ location of max and min

$\Rightarrow$ width of resonance $= \gamma$

zero's of ε_1 define $\omega_p^2 \equiv \dfrac{Ne^2}{m\varepsilon_0}$

$$0 = 1 + \omega_p^2\,\frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}$$

$$\Rightarrow (\omega^2-\omega_0^2)^2 - \omega_p^2(\omega^2-\omega_0^2) + \omega^2\gamma^2 = 0$$

For the zero near the resonance, $\omega^2\gamma^2 \rightarrow \omega_0^2\gamma^2$ a good approx

$\omega^2-\omega_0^2 \rightarrow (\Delta\omega)2\omega_0$, $\Delta\omega \equiv \omega-\omega_0$

$$(\Delta\omega)^2 4\omega_0^2 - \Delta\omega\, 2\omega_0\omega_p^2 + \omega_0^2\gamma^2 = 0$$

$$(\Delta\omega)^2 - \frac{\omega_p^2}{2\omega_0}\Delta\omega + \frac{\gamma^2}{4} = 0$$

for $\omega_p \gg \omega_0$, $\quad \Delta\omega \simeq \dfrac{\gamma^2\omega_0}{2\omega_p^2} = \dfrac{\gamma}{2}\underbrace{\left(\dfrac{\gamma}{\omega_0}\right)\left(\dfrac{\omega_0}{\omega_p}\right)^2}_{\text{both small}}$

generally true

shift of resonance small compared to width of resonance

For the zero above the resonance at ω_1

$$(\omega_1^2-\omega_0^2)^2 - \omega_p^2(\omega_1^2-\omega_0^2) + \underbrace{\omega_1^2\gamma^2}_{\text{small so ignore}} = 0$$

$$\Rightarrow \quad \omega_1^2-\omega_0^2 = \omega_p^2$$

$$\omega_1^2 = \omega_0^2+\omega_p^2 \simeq \omega_p^2 \quad \text{when } \omega_p \gg \omega_0$$

<u>max of</u> ε_2

$$\varepsilon_2 = \omega_p^2 \frac{\omega\gamma}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}$$

peak when $\frac{\partial \varepsilon_2}{\partial \omega} = 0 \Rightarrow \left((\omega_0^2-\omega^2)^2+\omega^2\gamma^2\right)\gamma - \omega\gamma\left[2(\omega_0^2-\omega^2)(-2\omega) + 2\omega\gamma^2\right] = 0$

$$\Rightarrow (\omega_0^2-\omega^2)^2\gamma + 4\omega^2\gamma(\omega_0^2-\omega^2) - \omega^2\gamma^3 = 0$$

near resonance,

$$(\omega_0^2-\omega^2) = \Delta\omega(2\omega_0) = \frac{\omega^2\gamma^3}{4\omega^2\gamma} = \frac{\gamma^2}{4}$$

$$\Delta\omega = \frac{\gamma^2}{8\omega_0} \quad \text{small} \Rightarrow \text{peak at} \simeq \omega_0$$

$$\frac{\varepsilon_2}{\varepsilon_0}(\omega_0) = \frac{\omega_p^2}{\omega\gamma}$$

half height at ω such that $\frac{\varepsilon_2}{\varepsilon_0}(\omega) = \frac{\omega_p^2}{2\omega\gamma}$

$$\Rightarrow \quad \frac{1}{2\omega\gamma} = \frac{\omega\gamma}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2} \Rightarrow (\omega_0^2-\omega^2)^2+\omega^2\gamma^2 = 2\omega^2\gamma^2$$

$$\omega_0^2-\omega^2 = \pm\omega\gamma$$

for <u>sharp</u> resonance $\Delta\omega(2\omega_0) = \pm\omega_0\gamma$

$$\Delta\omega \simeq \pm\frac{\gamma}{2}$$

width of resonance peak in $\frac{\varepsilon_2}{\varepsilon_0}$ is γ.

$$k = k_1 + i k_2 = \pm \frac{\omega}{c} \sqrt{\frac{\varepsilon_1}{\varepsilon_0} + i \frac{\varepsilon_2}{\varepsilon_0}}$$ if we want a wave traveling to the right, we take the + root,

want to express k_1 and k_2 in terms of ε_1 and ε_2

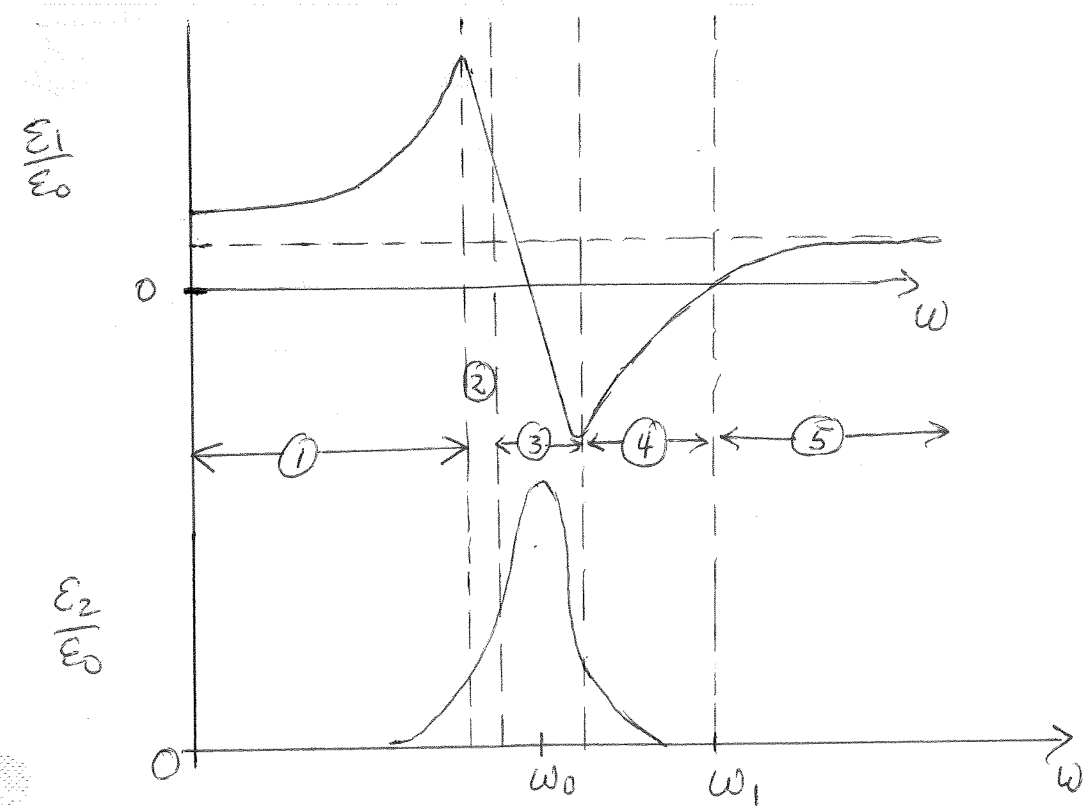
$$k^2 = k_1^2 - k_2^2 + 2 i k_1 k_2 = \frac{\omega^2}{c^2}\frac{\varepsilon_1}{\varepsilon_0} + i \frac{\omega^2}{c^2}\frac{\varepsilon_2}{\varepsilon_0}$$

equate real and imaginary pieces, and solve for k_1 and k_2

$$k_1 = \pm \frac{\omega}{c}\left[\frac{1}{2}\sqrt{\left(\frac{\varepsilon_1}{\varepsilon_0}\right)^2 + \left(\frac{\varepsilon_2}{\varepsilon_0}\right)^2} + \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)\right]^{1/2}$$

$$k_2 = \pm \frac{\omega}{c}\left[\frac{1}{2}\sqrt{\left(\frac{\varepsilon_1}{\varepsilon_0}\right)^2 + \left(\frac{\varepsilon_2}{\varepsilon_0}\right)^2} - \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)\right]^{1/2}$$

Regions of different behavior



Regions ① and ⑤ : *transparent propagation*

$\varepsilon_1 > 0 \qquad \varepsilon_1 \gg \varepsilon_2$

$$k_1 = \pm \frac{\omega}{c}\left[\frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)\sqrt{1+\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2} + \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)\right]^{1/2}$$

use $\sqrt{1+x} \simeq 1 + \frac{x}{2}$
small x

$$\simeq \pm \frac{\omega}{c}\left[\frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)\left(1+\frac{1}{2}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right) + \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\left[\frac{\varepsilon_1}{\varepsilon_0} + \frac{1}{4}\frac{\varepsilon_2^2}{\varepsilon_1\varepsilon_0}\right]^{1/2} \simeq \pm \frac{\omega}{c}\sqrt{\frac{\varepsilon_1}{\varepsilon_0}}\left(1 \pm O\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right)$$

small so can ignore

$$k_2 = \pm \frac{\omega}{c}\left[\frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)\left(1+\frac{1}{2}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right) - \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\left[\frac{1}{4}\frac{\varepsilon_2^2}{\varepsilon_1\varepsilon_0}\right]^{1/2} = k_1\left(\frac{\varepsilon_2}{2\varepsilon_1}\right)$$

so $k_2 \ll k_1$ small attenuation $\Rightarrow$ transparent propagation

index of refraction $n = \dfrac{ck_1}{\omega} \cong \sqrt{\dfrac{\varepsilon_1}{\varepsilon_0}}$

$\dfrac{dn}{d\omega} > 0 \Rightarrow$ <u>normal</u> dispersion ($v_g < v_p$)

phase velocity $v_p = \dfrac{\omega}{k_1} = \dfrac{c}{n} = c\sqrt{\dfrac{\varepsilon_0}{\varepsilon_1}}$

in region ① $\dfrac{\varepsilon_1}{\varepsilon_0} > 1 \Rightarrow v_p < c$

in region ⑤ $\dfrac{\varepsilon_1}{\varepsilon_0} < 1 \Rightarrow v_p > c$! (but $v_g < c$ always)

Region ②: Similar to region ①, except that $\frac{dn}{d\omega} < 0 \Rightarrow$ anomalous dispersion $(v_g > v_p)$

Region ③: $\omega \approx \omega_0$ resonant absorption

$$\frac{\varepsilon_2}{\varepsilon_1} = \frac{\omega_p^2}{\omega_0 \gamma} = \left(\frac{\omega_p}{\omega_0}\right)^2 \left(\frac{\omega_0}{\gamma}\right) \gg 1$$ for a sharp resonance $\frac{\gamma}{\omega_0} \ll 1$ (typically $\omega_p \gg \omega_0$)

So: $\varepsilon_2 \gg \varepsilon_1$

$$k_1 = \pm \frac{\omega}{c} \left[\frac{1}{2} \frac{\varepsilon_2}{\varepsilon_0} \sqrt{1 + \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2} + \frac{1}{2} \frac{\varepsilon_1}{\varepsilon_0} \right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c} \left[\frac{1}{2} \frac{\varepsilon_2}{\varepsilon_0} \left(1 + \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2\right) + \frac{1}{2} \frac{\varepsilon_1}{\varepsilon_0} \right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c} \left[\frac{1}{2} \frac{\varepsilon_2}{\varepsilon_0} + \frac{1}{4} \frac{\varepsilon_1^2}{\varepsilon_2 \varepsilon_0} + \frac{1}{2} \frac{\varepsilon_1}{\varepsilon_0} \right]^{1/2}$$

$$k_1 \simeq \pm \frac{\omega}{c} \sqrt{\frac{\varepsilon_2}{2\varepsilon_0}} \left(1 + O\left(\frac{\varepsilon_1}{\varepsilon_2}\right)\right)$$ small so can ignore

$$k_2 \simeq \pm \frac{\omega}{c} \left[\frac{1}{2} \frac{\varepsilon_2}{\varepsilon_0} + \frac{1}{4} \frac{\varepsilon_1^2}{\varepsilon_2 \varepsilon_0} - \frac{1}{2} \frac{\varepsilon_1}{\varepsilon_0} \right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c} \sqrt{\frac{\varepsilon_2}{2\varepsilon_0}}$$

$k_1 \simeq k_2 \Rightarrow$ strong attenuation

wave is exciting atoms near their resonant frequency ω_0 $\Rightarrow$ large atomic displacements $\Rightarrow$ media absorbs most energy from the wave. Wave decays rapidly (factor e^{-1}) within one wavelength of propagation.

Region ④ $\varepsilon_1 < 0$, $|\varepsilon_1| \gg \varepsilon_2$ total reflection

width of this region is $\omega_1 - \omega_0 = \sqrt{\omega_0^2 + \omega_p^2} - \omega_0 \sim \omega_p \sim \sqrt{N}$

width increases with atomic density since generally $\omega_p \gg \omega_0$

$$k_1 = \pm \frac{\omega}{c}\left[\frac{1}{2}\sqrt{\left(\frac{\varepsilon_1}{\varepsilon_0}\right)^2 + \left(\frac{\varepsilon_2}{\varepsilon_0}\right)^2} + \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)\right]^{1/2}$$

$$k_2 = \pm \frac{\omega}{c}\left[\frac{1}{2}\sqrt{\left(\frac{\varepsilon_1}{\varepsilon_0}\right)^2 + \left(\frac{\varepsilon_2}{\varepsilon_0}\right)^2} - \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)\right]^{1/2}$$

For $\varepsilon_1 < 0$ but $|\varepsilon_1| \gg \varepsilon_2$ we can write

$$k_1 = \pm \frac{\omega}{c}\left[\frac{1}{2}\left|\frac{\varepsilon_1}{\varepsilon_0}\right|\sqrt{1 + \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2} + \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)\right]^{1/2}$$

expand $\sqrt{\ }$

$$\simeq \frac{\omega}{c}\left[\frac{1}{2}\left|\frac{\varepsilon_1}{\varepsilon_0}\right|\left(1 + \frac{1}{2}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right) + \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)\right]^{1/2}$$

$$= \pm \frac{\omega}{c}\left[\frac{1}{2}\frac{|\varepsilon_1|}{\varepsilon_0} + \frac{1}{4}\frac{\varepsilon_2^2}{\varepsilon_0|\varepsilon_1|} + \frac{1}{2}\frac{\varepsilon_1}{\varepsilon_0}\right]^{1/2}$$

since $\varepsilon_1 < 0$ then $|\varepsilon_1| = -\varepsilon_1$ so

$$k_1 = \pm \frac{\omega}{c}\frac{1}{2}\frac{\varepsilon_2}{\sqrt{\varepsilon_0|\varepsilon_1|}} \simeq \pm \frac{\omega}{c}\frac{1}{2}\frac{\varepsilon_2}{|\varepsilon_1|}\sqrt{\frac{|\varepsilon_1|}{\varepsilon_0}}$$

whereas

$$k_2 \simeq \pm \frac{\omega}{c}\left[\frac{1}{2}\frac{|\varepsilon_1|}{\varepsilon_0} + \frac{1}{4}\frac{\varepsilon_2^2}{\varepsilon_0|\varepsilon_1|} - \frac{1}{2}\frac{\varepsilon_1}{\varepsilon_0}\right]^{1/2}$$

$\varepsilon_1 = -|\varepsilon_1|$

$$= \pm \frac{\omega}{c}\left[\frac{|\varepsilon_1|}{\varepsilon_0} + \frac{1}{4}\frac{\varepsilon_2^2}{\varepsilon_0|\varepsilon_1|}\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\sqrt{\frac{|\varepsilon_1|}{\varepsilon_0}}\left(1 + O\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right)$$

small so can ignore

$$\text{So } \frac{k_2}{k_1} = \frac{\sqrt{\frac{|\varepsilon_1|}{\varepsilon_0}}}{\frac{1}{2}\frac{\varepsilon_2}{|\varepsilon_1|}\sqrt{\frac{|\varepsilon_1|}{\varepsilon_0}}} = 2\frac{|\varepsilon_1|}{\varepsilon_2} \gg 1$$

wave vector k is almost pure imaginary $k_2 \gg k_1$
wave decays exponentially $\to 0$ before traveling even one wave length into the material

We will see that this is a region of <u>total reflection</u>.
Since $\omega \gg \omega_0$ in region (4), we are not at resonance, material is not absorbing much energy from the wave. The strong attenuation is due to the <u>destructive</u> <u>interference</u> between the wave and the induced fields of the polarized atoms.

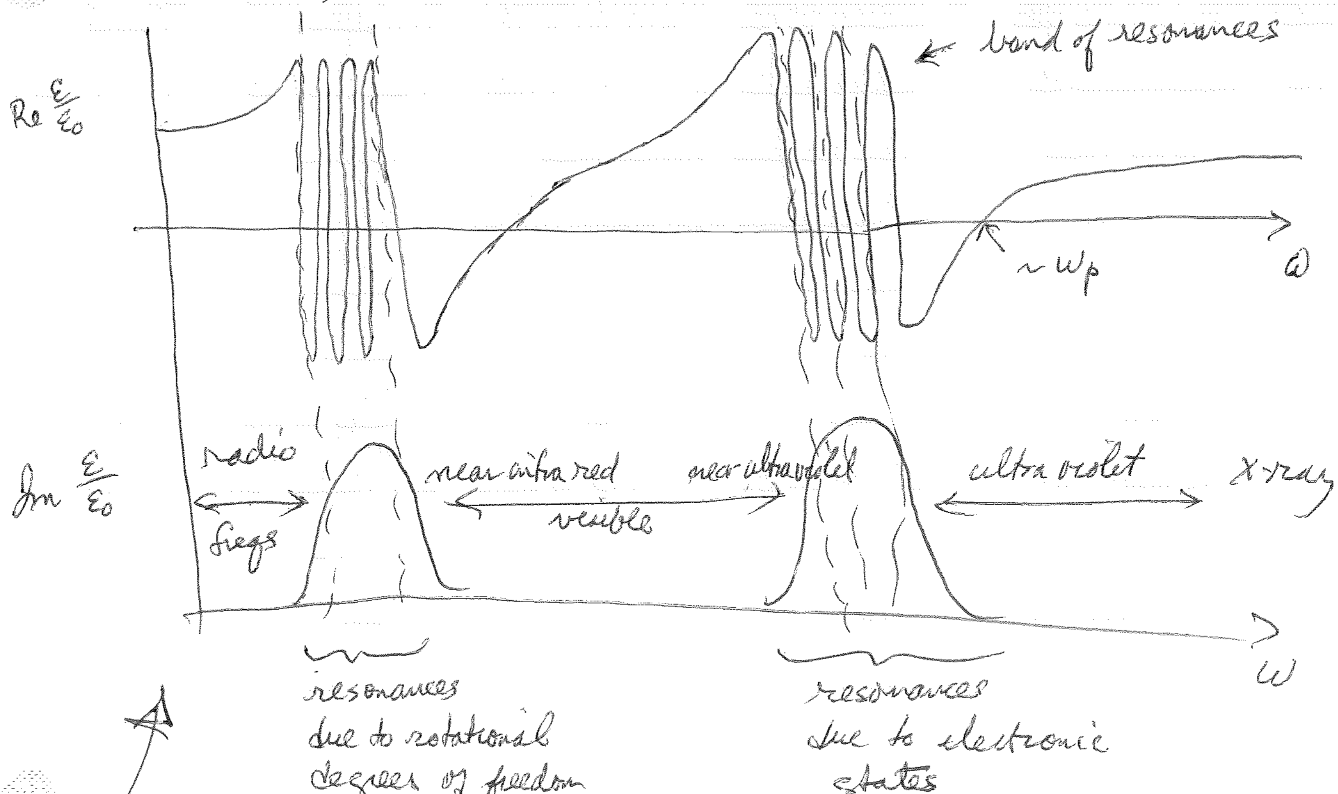
Our simple model was

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\gamma} \quad \leftarrow \text{single resonance at } \omega \approx \omega_0$$

A more realistic model of an atom or molecule would give many resonances

$$\varepsilon(\omega) = 1 + \omega_p^2 \sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

where $\hbar\omega_i$ are the energy spacings between quantized electron ~~et~~ energy levels with an allowed electric dipole transition.



for a typical molecular gas

$$\omega_p = \sqrt{\frac{Ne^2}{m\varepsilon_0}}$$

$$\omega_p = c\sqrt{\frac{N_A}{\varepsilon_0}\frac{e^2}{mc^2}}\sqrt{\frac{N}{N_A}}$$

$$\frac{e^2}{4\pi\varepsilon_0 mc^2} = 2.8\times10^{-13}\text{ cm}$$

$c = 3\times10^{10}$ cm/sec

$N_A = 6\times10^{23}$ cm^{-3} Avogadro's #

$$\omega_p = 4.4\times10^{16}\sqrt{\frac{N}{N_A}}\text{ sec}^{-1}$$

$$\hbar\omega_p = 185\sqrt{\frac{N}{N_A}}\text{ eV}$$

typical densities for H_2O or other dielectric liquid $\frac{N}{N_A} \simeq .05$

$\hbar\omega_p \simeq 40$ eV

compared to $\hbar\omega_0 \sim$ eV

for a metal, typical densities $\frac{N}{N_A} \simeq \frac{5\times10^{22}\text{ cm}^{-3}}{6\times10^{23}\text{ cm}^{-3}} \cong 0.1$

$\omega_p \simeq 10^{16}$ sec^{-1}

$\hbar\omega_p \simeq 58$ eV

For typical molecules, energies associated with electronic excitations are typically $\hbar\omega \sim O(1)$ eV. This sets the scale for ω_0, so we see, as claimed earlier, that generally $\omega_p \gg \omega_0$