

## Conductors

conduction electrons are free  $\rightarrow$  give  $\vec{j}_f$  and  $\rho_f$

$$m \ddot{\vec{r}} = -e \vec{E}(t) - \frac{m}{\tau} \dot{\vec{r}} \quad \tau \text{ is "collision time"}$$

$$\dot{\vec{r}} + \frac{\dot{\vec{r}}}{\tau} = -\frac{e}{m} \vec{E}$$

$$\vec{E} = \vec{E}_\omega e^{-i\omega t}$$

$$\Rightarrow \dot{\vec{r}} = \vec{v}_\omega e^{-i\omega t}$$

just like polarizable atom  
except  $\omega_0 = 0$  - no  
restoring force

$$(-\omega^2 + \frac{i\omega}{\tau}) \vec{v}_\omega = -\frac{e}{m} \vec{E}_\omega \Rightarrow \vec{v}_\omega = \frac{e}{m} \frac{1}{\omega^2 + \frac{i\omega}{\tau}} \vec{E}_\omega$$

$$= -\frac{e\tau}{m\omega} \frac{i}{1 - i\omega\tau} \vec{E}_\omega$$

current flow is  $\vec{j}_f = -eN\vec{v}$   
 $= -eN\dot{\vec{r}}$

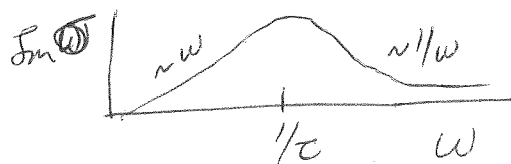
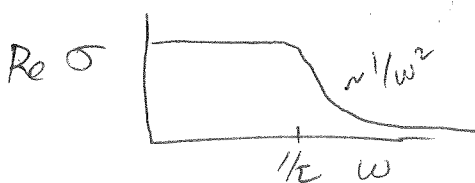
$N$  = density conduction  
electrons

$$\vec{j}_f = \vec{j}_\omega e^{-i\omega t}, \quad \vec{j}_\omega = -eN(-i\omega)\vec{v}_\omega$$

$$= \frac{Ne^2\tau}{m} \frac{1}{1 - i\omega\tau} \vec{E}_\omega$$

Define freq dependent conductivity

$$\vec{j}_\omega = \sigma(\omega) \vec{E}_\omega \Rightarrow \sigma(\omega) = \frac{Ne^2\tau}{m} \frac{1}{1 - i\omega\tau}$$



$$\text{Re } \sigma = \frac{\sigma_0}{1 + \omega^2\tau^2}$$

$$\text{Im } \sigma = \frac{\sigma_0\omega\tau}{1 + \omega^2\tau^2}$$

charge density obtained by charge conservation

$$\frac{\partial \rho_f}{\partial t} = -\vec{\nabla} \cdot \vec{j}_f$$

For a plane wave  $\vec{j}_f = \vec{j}_\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)}$   
 $\rho_f = \rho_\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)}$

$$-i\omega \rho_\omega = -i\vec{k} \cdot \vec{j}_\omega \Rightarrow \rho_\omega = \frac{\vec{k} \cdot \vec{j}_\omega}{\omega}$$

Maxwell's Equ  $\vec{E}(\vec{r}, t) = \vec{E}_\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)}$  etc.

$$1) \vec{\nabla} \cdot \vec{D} = \rho_{\text{free}}$$

$$2) \vec{\nabla} \cdot \vec{B} = 0$$

$$3) \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$4) \vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{j}_{\text{free}}$$

assume  $\vec{H} = \frac{\vec{B}}{\mu}$ ,  $\mu$  constant

$$\vec{D}_\omega = \epsilon_b(\omega) \vec{E}_\omega$$

$\epsilon_b(\omega)$  dielectric response from bound electrons

$$\vec{j}_\omega = \sigma(\omega) \vec{E}_\omega$$

$\sigma(\omega)$  conductivity due to free electrons

$$\rho_\omega = \frac{\vec{k} \cdot \vec{j}_\omega}{\omega} = \frac{\sigma(\omega)}{\omega} \vec{k} \cdot \vec{E}_\omega$$

$$1) \vec{\nabla} \cdot \vec{D} = \vec{P}_f \Rightarrow -i\vec{k} \cdot \vec{D}_\omega = \rho_\omega$$

$$\Rightarrow i\vec{k} \cdot \epsilon_b(\omega) \vec{E}_\omega = \frac{\sigma(\omega)}{\omega} \vec{k} \cdot \vec{E}_\omega$$

$$i\vec{k} \cdot \vec{E}_\omega \left[ \epsilon_b(\omega) + i \frac{\sigma(\omega)}{\omega} \right] = 0$$

$$2) i\vec{k} \cdot \mu \vec{H}_\omega = 0$$

$$3) i\vec{k} \times \vec{E}_\omega = i\omega \vec{B}_\omega = i\omega \mu \vec{H}_\omega$$

$$4) i\vec{k} \times \vec{H}_\omega = -i\omega \epsilon_b(\omega) \vec{E}_\omega + \sigma(\omega) \vec{E}_\omega$$

$$= -i\omega \left[ \epsilon_b(\omega) + i \frac{\sigma(\omega)}{\omega} \right] \vec{E}_\omega$$

Equations have exactly the same form as for waves in a dielectric provided we use

$$\epsilon(\omega) = \epsilon_b(\omega) + \frac{i\sigma(\omega)}{\omega}$$

transverse waves  
 $\vec{E} \perp \vec{k}$

and replace  $\mu_0$  by  $\mu$ .

dispersion relation for <sup>transverse</sup> waves is given by

$$k^2 = \omega^2 \mu \epsilon = \frac{\omega^2}{c^2} \frac{\mu}{\mu_0} \frac{\epsilon(\omega)}{\epsilon_0}$$

with  $\epsilon(\omega) = \epsilon_b(\omega) + \frac{i\sigma(\omega)}{\omega}$

[Note: for transverse mode,  $\vec{k} \perp \vec{E}_\omega$ , so  $\vec{k} \perp \vec{j}_\omega = \sigma(\omega) \vec{E}_\omega$   
 $\Rightarrow \rho_\omega = \frac{\vec{k} \cdot \vec{j}_\omega}{\omega} = 0$  no charge density oscillation!]

The main difference between wave propagation in dielectrics & conductors has to do with the contribution that the  $\frac{i\sigma(\omega)}{\omega}$  term makes to the real & imaginary parts of  $\epsilon(\omega)$

For our simple model (Drude model)

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau} \quad \text{where } \sigma_0 = \sigma(0) = \frac{Ne^2\tau}{m}$$

is d.c. conductivity

① Low frequencies  $\omega \ll 1/\tau$ ,  $\omega \ll \omega_0$   $\omega_0$  is resonant freq of  $\epsilon_b$

$$\epsilon_b(\omega) \approx \epsilon_b(0) \quad \text{real}$$

$$\sigma(\omega) \approx \sigma_0 \quad \text{real} \sim \tau$$

$$\frac{\epsilon(\omega)}{\epsilon_0} \approx \frac{\epsilon_b(0)}{\epsilon_0} + \frac{i\sigma_0}{\epsilon_0\omega}$$

← gives large imaginary part to  $\epsilon(\omega)$   
 grows as  $\frac{1}{\omega}$  as  $\omega \rightarrow 0$   
 $\Rightarrow$  strong dissipation

② High frequencies  $\omega \gg 1/\tau$ ,  $\omega \gg \omega_0$

$$\frac{\epsilon_b(\omega)}{\epsilon_0} \approx 1$$

$$\sigma(\omega) \approx \frac{\sigma_0}{-i\omega\tau} = \frac{iNe^2\tau}{m\omega\tau} = \frac{iNe^2}{m\omega}$$

imaginary part of  $\tau$

$$\frac{\epsilon(\omega)}{\epsilon_0} \approx 1 + \frac{i\sigma}{\epsilon_0\omega} \approx 1 - \frac{Ne^2}{\epsilon_0 m \omega^2} = \boxed{1 - \frac{\omega_p^2}{\omega^2} = \frac{\epsilon(\omega)}{\epsilon_0}}$$

where  $\omega_p = \sqrt{Ne^2/\epsilon_0 m}$  is "plasma freq" of conduction electrons

① Behavior at low freq

$$\frac{\epsilon(\omega)}{\epsilon_0} = \frac{\epsilon_b(\omega)}{\epsilon_0} + \frac{i\sigma_0}{\epsilon_0\omega} = \frac{\epsilon_b(\omega)}{\epsilon_0} \left( 1 + \frac{i\sigma_0}{\epsilon_b(\omega)\omega} \right)$$

Dissipation is due to  $\epsilon_2 = \text{Im } \epsilon$

Dissipation dominate when  $\epsilon_2 \gg \epsilon_1 = \text{Re } \epsilon$

ie when  $\frac{\sigma_0}{\epsilon_b(\omega)\omega} \gg 1$

this regime is called a "good" conductor - conduction electrons playing dominant role waves strongly attenuated

opposite limit:  $\frac{\sigma_0}{\epsilon_b(\omega)\omega} \ll 1$

this regime is called a "poor" conductor - waves propagate transparently - little relative absorption of energy from conduction electrons

One always gets into the "good" conductor limit as  $\omega$  decreases. For good conductor,

$$k \approx \frac{\omega}{c} \sqrt{\frac{\mu}{\mu_0} \frac{\epsilon'}{\epsilon_0}} \approx \frac{\omega}{c} \sqrt{\frac{\mu}{\mu_0}} \sqrt{i \frac{\epsilon_2}{\epsilon_0}} = \frac{\omega}{c} \sqrt{\frac{\mu}{\mu_0}} \sqrt{\frac{\sigma_0}{\epsilon_0\omega}} \sqrt{i}$$

$$k = \frac{\omega}{c} \sqrt{\frac{\mu}{\mu_0} \frac{\sigma_0}{\epsilon_0\omega}} \left( \frac{1+i}{\sqrt{2}} \right) \Rightarrow k_1 = k_2$$

real and imaginary parts of  $k$  are equal

$$\frac{1}{c} = \sqrt{\mu_0 \epsilon_0}$$

$$k_1 = k_2 = \frac{\omega}{c} \sqrt{\frac{\mu}{\epsilon_0 \mu_0} \frac{\sigma_0}{2\omega}} = \sqrt{\frac{\mu \sigma_0 \omega}{2}} \sim \sqrt{\omega}$$

Waves have form  $\vec{E} = E_0 e^{-k_2 z} e^{i(k_1 z - \omega t)}$

decay length of amplitude is

$$1/k_2 = \sqrt{\frac{2}{\mu \sigma_0 \omega}} = \delta \text{ "called the "skin depth" } \quad \text{Faraday case}$$

$\delta$  is distance wave penetrates into conductor

$\delta \sim 1/\sqrt{\omega}$  gets larger as  $\omega$  decreases

$$\vec{H} = \vec{H}_0 e^{-k_2 z} e^{i(k_1 z - \omega t + \phi)} \quad \left| \frac{\vec{H}_0}{\vec{E}_0} \right| = \frac{|k|}{\omega \mu}$$

phase shift between  $\vec{H}$  and  $\vec{E}$  is  $\phi$

$$\text{Given by } \tan \phi = k_2/k_1 \approx 1$$

$$\Rightarrow \phi \approx 45^\circ$$

$$\text{Amplitude ratio } \frac{|\vec{H}_0|}{|\vec{E}_0|} = \frac{|k|}{\omega \mu} = \frac{\sqrt{2} k_1}{\omega \mu} = \frac{\sqrt{2}}{\omega \mu} \sqrt{\frac{\mu \sigma_0 \omega}{2}}$$

$$= \sqrt{\frac{\sigma_0}{\omega \mu}} \text{ increases as } \frac{1}{\sqrt{\omega}} \text{ as } \omega \rightarrow 0$$

$\Rightarrow$  as  $\omega \rightarrow 0$ , most of energy of wave is carried by the magnetic field part of the wave.

## ② Behavior at high freq

$$\frac{\epsilon(\omega)}{\epsilon_0} \approx 1 - \left(\frac{\omega_p}{\omega}\right)^2 \quad \omega_p^2 = \frac{Ne^2}{m\epsilon_0} \quad \text{plasma freq}$$

$\epsilon(\omega)$  is real ( $\epsilon_2 \ll \epsilon_1$ )

1) If  $\omega > \omega_p$ , then  $\epsilon > 0$

$\Rightarrow$  transparent propagation  $k_1 = \frac{\omega}{c} \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$   
 $k$  is pure real  $k_2 \approx 0$

2) If  $\omega < \omega_p$ , then  $\epsilon < 0$

$\Rightarrow$  total reflection  $k_1 = 0$   
 $k_2 = \frac{\omega}{c} \sqrt{\frac{\mu}{\mu_0} \left(\frac{\omega_p^2}{\omega^2} - 1\right)}$   
 $k$  is pure imaginary

Plasma freq  $\omega_p$  gives cross over between reflection + transparent propagation.

$\tau \sim 10^{-14}$  sec for typical metal

$\omega_p \approx 10^{16}$  sec<sup>-1</sup> for most metals

$$\lambda_p = \frac{2\pi c}{\omega_p} \sim 3 \times 10^3 \text{ \AA}$$

(variable is  $\lambda \sim 5 \times 10^3 \text{ \AA}$ )

Example: The ionosphere is a layer of charged gas surrounding the earth. In many respects the charged gas behaves like conduction electrons in a metal. The plasma freq of the ionosphere is such that

for AM radio  $\omega_{AM} < \omega_p \Rightarrow$  AM radio reflected back to earth

for FM radio  $\omega_{FM} > \omega_p \Rightarrow$  FM radio propagates through ionosphere + escapes into space

Explains why you can pick up AM stations from far away - they are reflected back by ionosphere - but you only pick up local FM stations - they do not get reflected by ionosphere.