What about longitudinal modes? (ie H_ω, E_ω <u>not</u> $\perp \vec{k}$)

<u>magnetic field</u>

$i\mu \vec{k}\cdot\vec{H}_\omega = 0 \quad \Rightarrow \quad \vec{H}_\omega \perp \vec{k}$ $\underline{\underline{\text{or}}}$ $\vec{k}=0$ uniform magnetic field

Faraday $\quad \underbrace{i\vec{k}\times\vec{E}_\omega}_{\substack{\parallel \\ 0 \text{ as } \vec{k}=0}} = i\omega\mu \vec{H}_\omega \quad \Rightarrow \quad \omega = 0$

($\vec{H}\perp\vec{k}$ would be transverse mode so longitudinal mode must have $\vec{k}=0$ and so $\omega=0$.)

so only possible longitudinal magnetic field is a spatially uniform, constant in time $\vec{H}$.

<u>electric field</u>

$i\,\varepsilon(\omega)\,\vec{k}\cdot\vec{E}_\omega = 0 \quad \Rightarrow \quad \vec{E}_\omega \perp \vec{k}$, $\underline{\underline{\text{or}}}$ $\vec{k}=0$, $\underline{\underline{\text{or}}}$ $\varepsilon(\omega)=0$!

we can satisfy all Maxwell's equations for a $\vec{E}_\omega \parallel \vec{k}$, provided $\varepsilon(\omega)=0$, and by above, $\vec{H}_\omega = 0$ for this mode.

$i\vec{k}\times\vec{E}_\omega = i\omega\mu\vec{H}_\omega$ — both sides vanish.
LHS $=0$ as $\vec{E}_\omega \parallel \vec{k} \Rightarrow \vec{k}\times\vec{E}_\omega = 0$
RHS $=0$ as $\vec{H}_\omega = 0$

$i\vec{k}\times\vec{H}_\omega = -i\omega\varepsilon(\omega)\vec{E}_\omega$ — LHS $=0$ as $\vec{H}_\omega = 0$
RHS $=0$ as $\varepsilon(\omega)=0$

$i\mu\vec{k}\cdot\vec{H}_\omega = 0$ — satisfied as $\vec{H}_\omega = 0$

So we can have a longitudinal ~~oscillation~~ $\vec{E}$ provided $\varepsilon(\omega)=0$

frequencies of longitudinal mode given by $\varepsilon(\omega)=0$.

<u>low freq</u> $\omega \ll \omega_0$, $\omega\tau \ll 1$ $\quad$ N_a = density of polarizable atoms

$$\frac{\varepsilon}{\varepsilon_0} = \frac{\varepsilon_b}{\varepsilon_0} + \frac{i\sigma}{\varepsilon_0\omega} \simeq \overbrace{1 + \frac{N_a e^2}{m\varepsilon_0}}^{\varepsilon_b(0)/\varepsilon_0} + \frac{i\sigma_0}{\varepsilon_0\omega} = \frac{1}{\varepsilon_0}\left(\varepsilon_b(0) + \frac{i\sigma_0}{\omega}\right)$$

$$\frac{\varepsilon}{\varepsilon_0} = 0 \quad \text{when} \quad \omega = \frac{-i\sigma_0}{\varepsilon_b(0)}$$

$$\Rightarrow \vec{E}(\vec{r},t) = \vec{E}_\omega e^{i(\vec{k}\cdot\vec{r} - \omega t)} = \vec{E}_\omega e^{-\sigma_0 t/\varepsilon_b(0)} e^{i\vec{k}\cdot\vec{r}}$$

$\Rightarrow$ if set up a longitudinal $\vec{E}$ field, it decays to zero exponentially fast, with decay time $\frac{\varepsilon_b(0)}{\sigma_0}$

Consistant with our assumption that $\vec{E}=0$ inside a conductor for electrostatics.

(electrostatic fields are <u>always</u> longitudinal

$$\vec{E} = -\vec{\nabla}V \Rightarrow \cancel{\vec{E}_k = -i\vec{k}V_k \text{ for Fourier component}}$$

$$\vec{E} \sim -i\vec{k}V_k e^{i\vec{k}\cdot\vec{r}} \qquad \vec{E} \sim \vec{k}$$

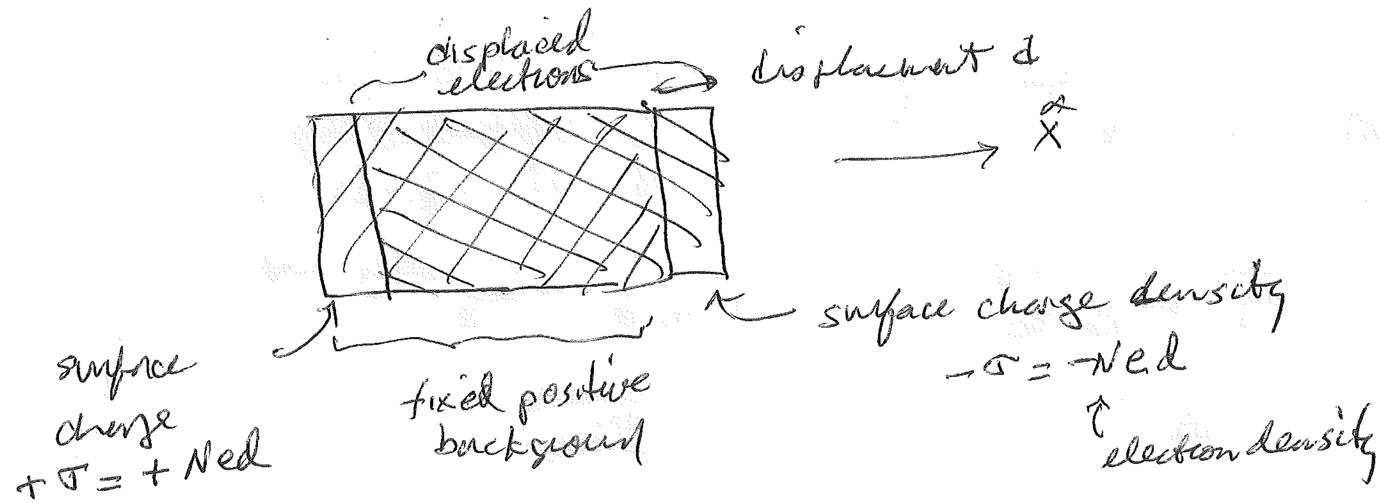
<u>high freq</u> $\omega \gg 1/\tau$, $\omega \gg \omega_0$ $\quad$ N = density of conduction electrons

then $\dfrac{\varepsilon(\omega)}{\varepsilon_0} = 1 - \dfrac{\omega_p^2}{\omega^2}$ $\qquad \omega_p^2 = \dfrac{Ne^2}{\varepsilon_0 m}$

$\varepsilon(\omega) = 0$ when $\omega = \omega_p$ the plasma freq

longitudinal oscillation of $\vec{E}$ (and ρ) at $\omega = \omega_p$

simple model for longitudinal mode



electric field $\vec{E} = \dfrac{\sigma}{\varepsilon_0}\hat{x}$

force per volume on electrons is $-NeE\hat{x}$

Newtons equ: $mN\dfrac{d^2 d}{dt^2} = -NeE\hat{x}$

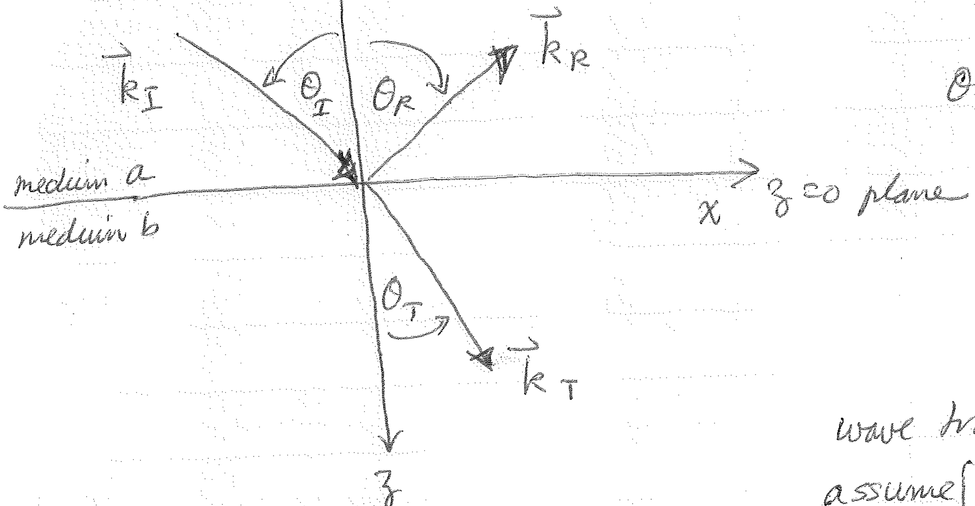
gives oscillating solution

plasma oscillation
- oscillating charge density
- oscillating electric field

$$\vec{\nabla}\cdot\vec{E} \rightarrow i\vec{k}\cdot\vec{E}_\omega = \frac{\rho_\omega}{\varepsilon_0} \neq 0$$

$\Rightarrow$ longitudinal mode for $\vec{E}$ induces a charge density oscillation at same frequency ω_p

Reflection and Transmission (Refraction) of waves



θ_I = angle of incidence
θ_R = angle of reflection
θ_T = angle of transmission (refraction)

wave traveling from a to b.

assume $\begin{cases} \mu_a \text{ and } \mu_b \text{ are real} \\ \varepsilon_a \text{ real} \\ \varepsilon_b \text{ may be complex} \end{cases}$

$$\vec{E}_I = \vec{E}_{\omega I}\, e^{i(\vec{k}_I\cdot\vec{r} - \omega_I t)}$$

$$\vec{E}_R = \vec{E}_{\omega R}\, e^{i(\vec{k}_R\cdot\vec{r} - \omega_R t)}$$

$$\vec{E}_T = \vec{E}_{\omega T}\, e^{i(\vec{k}_T\cdot\vec{r} - \omega_T t)}$$

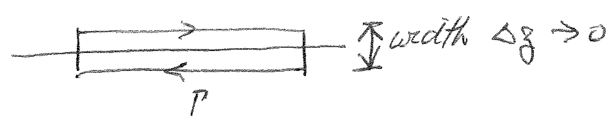
similarly for $\vec{H}_I,\ \vec{H}_R,\ \vec{H}_T$

in each media $k^2 = \frac{\omega^2}{c^2}\frac{\mu}{\mu_0}\frac{\varepsilon}{\varepsilon_0} = \omega^2\mu\varepsilon$

$$k_I^2 = \omega_I^2 \mu_a \varepsilon_a\ ,\quad k_R^2 = \omega_R^2 \mu_a \varepsilon_a\ ,\quad k_T^2 = \omega_T^2 \mu_b \varepsilon_b$$

boundary conditions at interface

Faraday $\quad \vec{\nabla}\times\vec{E}_\omega - i\omega\mu\vec{H}_\omega = 0$



$$\int_S d\vec{a}\cdot(\vec{\nabla}\times\vec{E}_\omega) = \int_S d\vec{a}\cdot\vec{H}_\omega\, i\omega\mu \to 0 \quad \text{as } \Delta z \to 0$$

surface bounded by Γ

$$\oint_\Gamma d\vec{l}\cdot\vec{E}_\omega \Rightarrow (\vec{E}_{above} - \vec{E}_{below})\cdot d\vec{l} = 0$$

$\Rightarrow$ tangential component of $\vec{E}$ is continuous as across interface

Ampere $\quad \nabla \times \vec{H}_\omega = -i\omega\varepsilon \vec{E}_\omega$ $\quad$ assuming no free current except that ~~[illegible]~~ ~~[illegible]~~ due to conduction electrons

same arguement as for $\vec{E}$ $\Rightarrow$ tangential component of $\vec{H}$ is continuous at interface

apply to $\vec{E}$ at interface: For $\hat{t}$ any unit vector in xy plane

$$\hat{t} \cdot (\vec{E}_I + \vec{E}_R) = \hat{t} \cdot \vec{E}_T$$

$\Rightarrow$ for any $\vec{\rho}$ in xy plane at $z=0$, and any time t

$$\hat{t} \cdot \vec{E}_{\omega I}\, e^{i(\vec{k}_I \cdot \vec{\rho} - \omega_I t)} + \hat{t} \cdot \vec{E}_{\omega R}\, e^{i(\vec{k}_R \cdot \vec{\rho} - \omega_R t)}$$

$$= \hat{t} \cdot \vec{E}_{\omega T}\, e^{i(\vec{k}_T \cdot \vec{\rho} - \omega_T t)}$$

true for any $\vec{\rho}$, so consider at $\vec{\rho} = 0$

$$\hat{t} \cdot \vec{E}_{\omega I}\, e^{-i\omega_I t} + \hat{t} \cdot \vec{E}_{\omega R}\, e^{-i\omega_R t} = \hat{t} \cdot \vec{E}_{\omega T}\, e^{-i\omega_T t}$$

must be true for all t $\Rightarrow$ $\boxed{\omega_I = \omega_R = \omega_T}$
all freq's equal

Now consider for $\rho \neq 0$, at $t = 0$.

$$\hat{t} \cdot \vec{E}_{\omega I}\, e^{i\vec{k}_I \cdot \vec{\rho}} + \hat{t} \cdot \vec{E}_{\omega R}\, e^{i\vec{k}_R \cdot \vec{\rho}} = \hat{t} \cdot \vec{E}_{\omega T}\, e^{i\vec{k}_T \cdot \vec{\rho}}$$

must be true for <u>all</u> $\vec{\rho}$ $\Rightarrow$ $\vec{k}_I \cdot \vec{\rho} = \vec{k}_R \cdot \vec{\rho} = \vec{k}_T \cdot \vec{\rho}$ all ρ

$\Rightarrow$ projections of $\vec{k}_I$, $\vec{k}_R$, $\vec{k}_T$ in xy plane are all equal.
only z-components of $\vec{k}_I$, $\vec{k}_R$, $\vec{k}_T$ may differ

choose coordinates as in diagram so that all $\vec{k}$'s lie in xz plane.

$$k_{Ix} = k_{Rx} \Rightarrow |\vec{k}_I| \sin\theta_I = |\vec{k}_R| \sin\theta_R$$

$$|\vec{k}_I| = \omega\sqrt{\mu_a \varepsilon_a} = |\vec{k}_R| \Rightarrow \boxed{\theta_I = \theta_R}$$

<u>angle of incidence = angle of reflection</u>

If $\sqrt{\varepsilon_b}$ is also real (ie in region of transparent propagation)

then $|\vec{k}_T| = \omega\sqrt{\mu_b \varepsilon_b}$

$$k_{Ix} = k_{Tx} \Rightarrow |\vec{k}_I| \sin\theta_I = |\vec{k}_T| \sin\theta_T$$

$$\omega\sqrt{\mu_a \varepsilon_a} \sin\theta_I = \omega\sqrt{\mu_b \varepsilon_b} \sin\theta_T$$

$$\frac{\sin\theta_T}{\sin\theta_I} = \sqrt{\frac{\mu_a \varepsilon_a}{\mu_b \varepsilon_b}}$$

in terms of index of refraction $n \equiv \frac{kc}{\omega} = \frac{\omega\sqrt{\mu\varepsilon}\, c}{\omega}$

$n \equiv \frac{c}{v_p}$ $\qquad$ $n = \sqrt{\mu\varepsilon}\, c = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$

$$\boxed{\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_a}{n_b}}$$

Snell's law – true for all types of waves, not just EM waves

$$\sin\theta_T = \frac{n_a}{n_b}\sin\theta_I$$

If $n_a > n_b$, then $\theta_T > \theta_I$

in this case,

when θ_I is too large, we will have $\frac{n_a}{n_b}\sin\theta_I > 1$

and there is no solution for θ_T

$\Rightarrow \vec{E}_T = 0$, there is no transmitted wave.

this is called "total internal reflection" – wave does not exit medium a.

critical angle $\theta_c = \arcsin\left(\frac{n_b}{n_a}\right) \leftarrow$ [the bigger n_a/n_b, the smaller θ_c

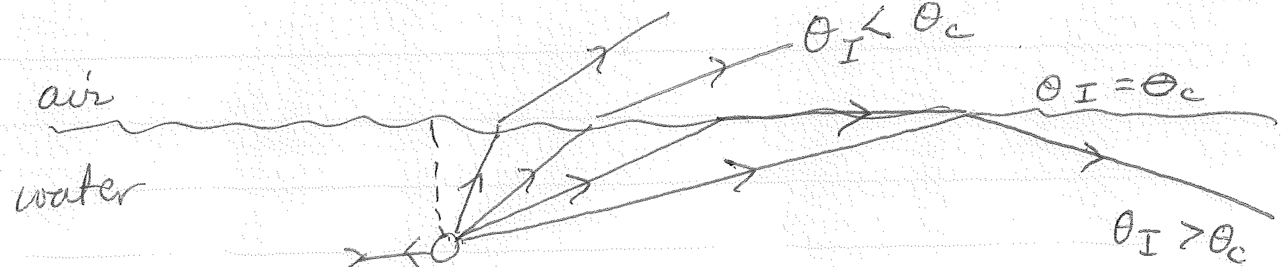
total internal reflection whenever $\theta_I > \theta_c$

total internal reflection usually happens as one goes from a ~~denser~~ denser to a less dense ~~material~~ medium as

$\left(\frac{n}{c}\right)^2 = \mu\epsilon \sim \mu\epsilon_0\left(1 + \frac{Ne^2}{m\epsilon_0}\right)$ where N is density of polarizable atoms (m is electron mass).

total internal reflection is why diamonds sparkle! diamond has big $n \Rightarrow$ small $\theta_c \Rightarrow$ light bounces around inside diamond getting totally internally reflected many times, before it is able to escape.

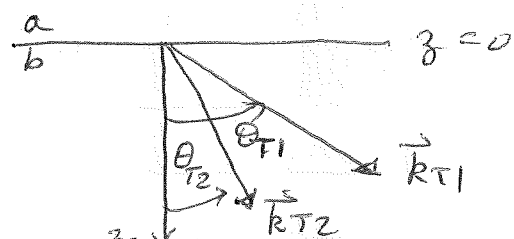
Can also experience total internal reflection in the swimming pool:



$n_{water} > n_{air}$ swimmer for $\theta_I > \theta_c$, can't see out of water

when $\theta_I = \theta_c$, transmitted wave travels parallel to interface

<u>More general case</u>: $\sqrt{\varepsilon_b}$ can be complex $\Rightarrow \vec{k}_T$ is complex



$\vec{k}_T = \vec{k}_{T1} + i\vec{k}_{T2}$

$k_{T1} \equiv |\vec{k}_{T1}|$, $\vec{k}_{T2} \equiv |\vec{k}_{T2}|$

$\vec{k}_{T1}$ and $\vec{k}_{T2}$ need not be in same direction!

$$\vec{k}_{TX} = \vec{k}_{IX} \Rightarrow k_{T1} \sin\theta_{T1} + i k_{T2} \sin\theta_{T2} = k_I \sin\theta_I$$

equate real and imaginary pieces $\Rightarrow$ $\boxed{\begin{array}{l} k_{T1} \sin\theta_{T1} = k_I \sin\theta_I \\ k_{T2} \sin\theta_{T2} = 0 \end{array}}$

$\Rightarrow \boxed{\theta_{T2} = 0}$

ie attenuation factor for the transmitted wave is of the form $e^{-k_{T2} z}$

$\Rightarrow$ planes of constant <u>amplitude</u> are <u>parallel</u> to the interface, no matter what the angle of incidence θ_I.