planes of constant phase are $\perp$ to $\vec{k}_{T1}$

Now we solve for k_{T1} and k_{T2} and θ_{T1}

Dispersion relation in medium 2: $k_T^2 = \omega^2 \mu_b \varepsilon_b$

$$k_T^2 = (\vec{k}_{T1} + i\vec{k}_{T2})^2 = k_{T1}^2 - k_{T2}^2 + 2i\vec{k}_{T1}\cdot\vec{k}_{T2}$$
$$= k_{T1}^2 - k_{T2}^2 + 2i k_{T1} k_{T2} \cos\theta_{T1} \qquad (\text{since } \theta_{T2}=0)$$
$$= \omega^2 \mu_b (\varepsilon_{b1} + i\,\varepsilon_{b2})$$

equate real and imaginary parts of both sides

$$\left.\begin{aligned} k_{T1}^2 - k_{T2}^2 &= \omega^2 \mu_b \varepsilon_{b1} \\ 2k_{T1}k_{T2} &= \frac{\omega^2 \mu_b \varepsilon_{b2}}{\cos\theta_{T1}} \end{aligned}\right\}$$

same equations as when we considered propagation in an infinite dielectric, only then $\theta_{T1}=0$

Consider the above as two equations for two unknowns k_{T1} and k_{T2}. Solve for k_{T1} and k_{T2} in terms of $\cos\theta_{T1}$

1) $$k_{T1} = \omega\sqrt{\mu_b}\left[\frac{1}{2}\sqrt{\varepsilon_{b1}^2 + \frac{\varepsilon_{b2}^2}{\cos^2\theta_{T1}}} + \frac{\varepsilon_{b1}}{2}\right]^{1/2}$$

2) $$k_{T2} = \omega\sqrt{\mu_b}\left[\frac{1}{2}\sqrt{\varepsilon_{b1}^2 + \frac{\varepsilon_{b2}^2}{\cos^2\theta_{T1}}} - \frac{\varepsilon_{b1}}{2}\right]^{2}$$

If $\theta_{T1}=0$, this is the same as our earlier result

Finally we use our boundary condition to determine θ_{T1}

3) $$k_{T1}\sin\theta_{T1} = k_I \sin\theta_I$$

solve (1)+(2)+(3) for three unknowns k_{T1}, k_{T2}, θ_{T1}

planes of constant <u>phase</u> are $\perp$ to $\vec{k}_{T1}$

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same equations as when we considered propagation in an infinite dielectric, only then $\theta_{T1} = 0$

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2) $$k_{T2} = \omega\sqrt{\mu_b}\left[\frac{1}{2}\sqrt{\varepsilon_{b1}^2 + \frac{\varepsilon_{b2}^2}{\cos^2\theta_{T1}}} - \frac{\varepsilon_{b1}}{2}\right]^{2}$$

If $\theta_{T1} = 0$, this is the same as our earlier result

Finally we use our boundary condition to determine θ_{T1}

3) $$k_{T1} \sin\theta_{T1} = k_I \sin\theta_I$$

solve (1) + (2) + (3) for three unknowns k_{T1}, k_{T2}, θ_{T1}

$$k_I = \omega\sqrt{\mu_a \epsilon_a} = \frac{\omega}{c}\sqrt{\frac{\mu_a \epsilon_a}{\mu_0 \epsilon_0}} = \frac{\omega}{c} n_a$$

← index of refraction

(3) ⇒ $$k_{T1} = \frac{k_I \sin\theta_I}{\sin\theta_{T1}} = \frac{\omega}{c}\frac{n_a \sin\theta_I}{\sin\theta_{T1}}$$

(1) $$= \omega\sqrt{\mu_b}\left[\frac{1}{2}\sqrt{\epsilon_{b1}^2 + \frac{\epsilon_{b2}^2}{\cos^2\theta_{T1}}} + \frac{\epsilon_{b1}}{2}\right]^{1/2}$$

$$\Rightarrow \boxed{n_a \sin\theta_I = c\sqrt{\mu_b}\left[\frac{1}{2}\sqrt{\epsilon_{b1}^2 + \frac{\epsilon_{b2}^2}{\cos^2\theta_{T1}}} + \frac{\epsilon_{b1}}{2}\right]^{1/2}\sin\theta_{T1}}$$

↑ determines angle of transmission θ_{T1} in terms of angle of incidence θ_I and the physical parameters $n_a, \mu_b, \epsilon_{b1}, \epsilon_{b2}$ of the two materials

Cases ① If material b is nearly transparent, ie $\epsilon_{b2} \ll \epsilon_{b1}$

define $n_b = \sqrt{\frac{\mu_b \epsilon_{b1}}{\mu_0 \epsilon_0}} = \sqrt{\mu_b \epsilon_{b1}}\, c$

then $$n_a \sin\theta_I = n_b \sin\theta_{T1}\left[\frac{1}{2\epsilon_{b1}}\sqrt{\epsilon_{b1}^2 + \frac{\epsilon_{b2}^2}{\cos^2\theta_{T1}}} + \frac{1}{2}\right]^{1/2}$$

$$= n_b \sin\theta_{T1}\left[\frac{1}{2}\sqrt{1 + \frac{\epsilon_{b2}^2}{\epsilon_{b1}^2\cos^2\theta_{T1}}} + \frac{1}{2}\right]^{1/2}$$

expand the $\sqrt{1+\delta} \approx 1 + \delta/2$

$$\simeq n_b \sin\theta_{T1}\left[\frac{1}{2} + \frac{\epsilon_{b2}^2}{4\epsilon_{b1}^2\cos^2\theta_{T1}} + \frac{1}{2}\right]^{1/2}$$

$$= n_b \sin\theta_{T1}\left[1 + \frac{\epsilon_{b2}^2}{4\epsilon_{b1}^2\cos^2\theta_{T1}}\right]^{1/2}$$

expand the $\sqrt{}$

$$n_a \sin\theta_I \simeq n_b \sin\theta_{T1}\left[1 + \underbrace{\frac{\varepsilon_{b2}^2}{8\varepsilon_{b1}^2 \cos^2\theta_{T1}}}_{\text{small correction to Snell's Law}}\right]$$

when $\frac{\varepsilon_{b2}}{\varepsilon_{b1}} \ll 1$, we can solve above equation iteratively to get approximate result

$$n_a \sin\theta_I = n_b \sin\theta_{T1}\left[1 + \text{small}\right]$$

$$\Rightarrow \sin\theta_{T1} \approx \frac{n_a}{n_b}\sin\theta_I \quad \Rightarrow \cos^2\theta_{T1} \approx 1 - \frac{n_a^2}{n_b^2}\sin^2\theta_I$$

so to next order

$$n_b \sin\theta_{T1} \simeq \frac{n_a \sin\theta_I}{1 + \frac{1}{8}\left(\frac{\varepsilon_{b2}}{\varepsilon_{b1}}\right)^2\left[\frac{1}{1 - \frac{n_a^2}{n_b^2}\sin^2\theta_I}\right]}$$

$$\simeq n_a \sin\theta_I\left[1 - \underbrace{\frac{1}{8}\left(\frac{\varepsilon_{b2}}{\varepsilon_{b1}}\right)^2 \frac{1}{1 - \frac{n_a^2}{n_b^2}\sin^2\theta_I}}_{\text{this term is } > 0 \text{ so} \ldots}\right]$$

$$\lesssim n_a \sin\theta_I$$

Result is that θ_{T1} is <u>smaller</u> than one would predict from Snell's law.

The correction is of order $O\left(\frac{\varepsilon_{b2}}{\varepsilon_{b1}}\right)^2$.

medium b is a

Case (2) good conductor or a region of resonant absorption of a dielectric so $\varepsilon_{b2} \gg \varepsilon_{b1}$

Now, to lowest order we will approx $\varepsilon_{b1} \approx 0$ then

$$n_a \sin\theta_I = c\sqrt{\mu_b}\left[\frac{1}{2}\frac{\varepsilon_{b2}}{\cos\theta_{T1}}\right]^{1/2} \sin\theta_{T1}$$

$$\boxed{n_a \sin\theta_I = c\sqrt{\frac{\mu_b \varepsilon_{b2}}{2}}\;\frac{\sin\theta_{T1}}{\sqrt{\cos\theta_{T1}}}}$$

determines θ_{T1} in terms of θ_I

In this case our result for θ_{T1} looks nothing like Snell's Law.

$\Rightarrow$ Snell's Law only holds if both media are transparent at the frequency of interest

So far, all our results come from the requirement that the phases of the incident, reflected, and transmitted waves all match at the interface. This is enough to determine the directions, wavelengths, attenuation, and frequencies of the waves. These results hold for <u>any</u> type of wave, not just electromagnetic waves.

Now want to solve for amplitudes of transmitted and reflected waves.

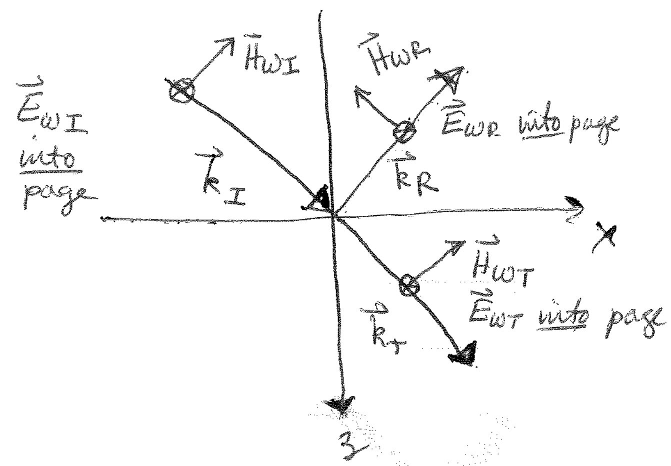
<u>two cases:</u> "plane of incidence" = plane spanned by the wavevectors $\vec{k}_I$, and the normal to the interface — in our case, the xz plane

① $\vec{E}_\omega$ is $\perp$ to the plane of incidence

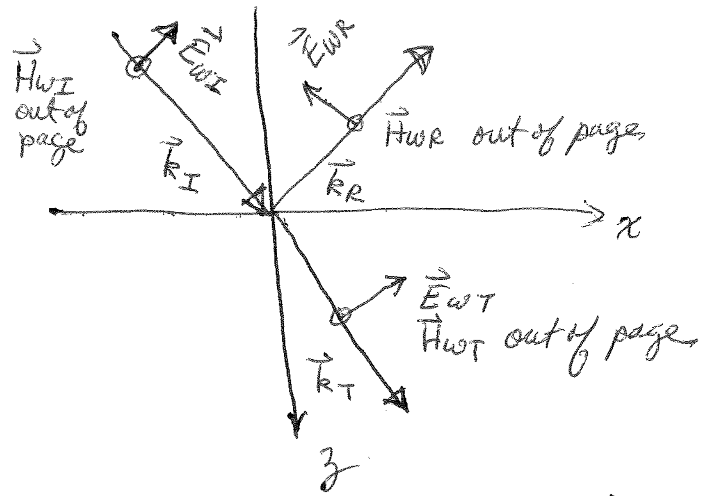
② $\vec{E}_\omega$ is $\parallel$ to the plane of incidence

The most general case is a linear superposition of these two, so treating these two cases separately also gives the general solution.

$E_0 \perp$ plane of incidence	$E_0 \parallel$ plane of incidence



($H_{\omega I}$ in plane of incidence)



($H_{\omega I} \perp$ to plane of incidence)

continuity of $\hat{y}$ components of $\vec{E}$ | of $\vec{H}$

all the $\vec{E}$'s are along $\hat{y}$

(1) $E_I + E_R = E_T$

where $\vec{E}_{\omega I} = E_I \hat{y}$ etc.

all the $\vec{H}$'s are along $\hat{y}$

(1) $H_I + H_R = H_T$

where $\vec{H}_{\omega I} = H_I \hat{y}$ etc.

continuity of $\hat{x}$ components of $\vec{H}$ | of $\vec{E}$

$$H_{Ix} + H_{Rx} = H_{Tx}$$

Faraday $\quad H_x = \dfrac{k_z}{\omega\mu} E_y$

$i\mu\omega \vec{H} = i\vec{k}\times\vec{E}$

plug in above and use $k_{Iz} = -k_{Rz}$

$\Rightarrow$

$$(2) \quad \frac{k_{Iz}}{\mu_a}(E_I - E_R) = \frac{k_{Tz}}{\mu_b} E_T$$

solve equation (1) and (2) for E_R and E_T in terms of E_I

$$E_{Ix} + E_{Rx} = E_{Tx}$$

Ampere $\quad E_x = \dfrac{-k_z}{\omega\varepsilon} H_y$

$-i\omega\varepsilon\vec{E} = i\vec{k}\times\vec{H}$

plug in above and use $k_{Iz} = -k_{Rz}$

$\Rightarrow$

$$(2) \quad \frac{k_{Iz}}{\varepsilon_a}(H_I - H_R) = \frac{k_{Tz}}{\varepsilon_b} H_T$$

solve equations (1) and (2) for H_R and H_T in terms of H_I

$$E_R = \frac{\mu_b k_{Iz} - \mu_a k_{Tz}}{\mu_b k_{Iz} + \mu_a k_{Tz}} E_I$$

$$E_T = \frac{2\mu_b k_{Iz}}{\mu_a k_{Tz} + \mu_b k_{Iz}} E_I$$

$$H_R = \frac{\varepsilon_b k_{Iz} - \varepsilon_a k_{Tz}}{\varepsilon_b k_{Iz} + \varepsilon_a k_{Tz}} H_I$$

$$H_T = \frac{2\varepsilon_b k_{Iz}}{\varepsilon_a k_{Tz} + \varepsilon_b k_{Iz}} H_I$$

We can now define the reflection and transmission coefficients. These are defined in terms of the transported energy. Since the energy flux $|\vec{S}|$ is $\sim |\vec{E}|^2 \sim |\vec{H}|^2$, we have

<u>Reflection coefficient</u>

① $E_0 \perp$ to plane of incidence

$$R_\perp = \frac{|E_R|^2}{|E_I|^2} = \left| \frac{\mu_b k_{Iz} - \mu_a k_{Tz}}{\mu_b k_{Iz} + \mu_a k_{Tz}} \right|^2$$

② $E_0 \parallel$ to plane of incidence

$$R_\parallel = \frac{|H_R|^2}{|H_I|^2} = \left| \frac{\varepsilon_b k_{Iz} - \varepsilon_a k_{Tz}}{\varepsilon_b k_{Iz} + \varepsilon_a k_{Tz}} \right|^2$$

For region of "total reflection" in material b, $\operatorname{Im}\varepsilon_b \simeq 0$, $\operatorname{Re}\varepsilon_b < 0$

$\Rightarrow \vec{k}_T \equiv i\vec{K}_T$ where $\vec{K}_T$ is real ($\vec{k}_T$ is pure imaginary)

$$\left.\begin{aligned} \Rightarrow R_\perp &= \left| \frac{\mu_b k_{Iz} - i\mu_a K_{Tz}}{\mu_b k_{Iz} + i\mu_a K_{Tz}} \right|^2 \\ R_\parallel &= \left| \frac{\varepsilon_b k_{Iz} - i\varepsilon_a K_{Tz}}{\varepsilon_b k_{Iz} + i\varepsilon_a K_{Tz}} \right|^2 \end{aligned}\right\}$$

both are of the form $\left| \frac{a - ib}{a + ib} \right|^2 = 1$ when a, b both real