Radiation by localized oscillating charge distribution

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{t} dt' \, \vec{J}(\vec{r}',t') \, \overbrace{\frac{\delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|}}^{G(\vec{r}-\vec{r}',\, t-t')}$$

For pure harmonic oscillating current $\vec{J}(\vec{r},t) = Re\{\vec{J}(r,\omega)\, e^{-i\omega t}\}$

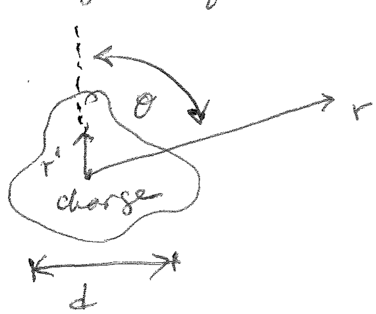
Resulting $\vec{A}$ will oscillate at same freq ω $\vec{A}(r,t) = Re\{\vec{A}(\vec{r},\omega)\, e^{-i\omega t}\}$

$$\vec{A}(\vec{r},\omega)\, e^{-i\omega t} = \frac{\mu_0}{4\pi}\int d^3r' \, dt' \, \vec{J}(\vec{r}',\omega)\, e^{-i\omega t'} \frac{\delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|}$$

$$\vec{A}(\vec{r},\omega)\, e^{-i\omega t} = \frac{\mu_0}{4\pi}\int d^3r' \, \vec{J}(\vec{r}',\omega)\, e^{-i\omega t} \frac{e^{+i\omega\frac{|\vec{r}-\vec{r}'|}{c}}}{|\vec{r}-\vec{r}'|}$$

$$\vec{A}(\vec{r},\omega) = \frac{\mu_0}{4\pi}\int d^3r' \, \vec{J}(\vec{r}',\omega) \frac{e^{i\omega|\vec{r}-\vec{r}'|/c}}{|\vec{r}-\vec{r}'|}$$

Assume $\vec{J}(\vec{r}',\omega) \simeq 0$ for $|\vec{r}'| > d$, ie charge is localized with region of size d about origin.



Approx ①

for $r \gg d$, ie far from sources

$$|\vec{r}-\vec{r}'| = \sqrt{r^2 + r'^2 - 2rr'\cos\theta}$$

$$= r\sqrt{1 + \left(\frac{r'}{r}\right)^2 - \frac{2r'}{r}\cos\theta}$$

$$\simeq r\left(1 - \frac{r'}{r}\cos\theta + O\left(\frac{r'}{r}\right)^2\right)$$

$$\simeq r - \vec{r}' \cdot \hat{r}$$

$\hat{r}$ is unit vector along $\vec{r}$

$$\vec{A}(\vec{r},\omega) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\vec{J}(\vec{r}',\omega)\, e^{ik(r-\vec{r}'\cdot\hat{r})}}{r-\vec{r}'\cdot\hat{r}} \qquad \text{where } k \equiv \frac{\omega}{c}$$

$$= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r' \frac{\vec{J}(\vec{r}',\omega)\, e^{-ik\vec{r}'\cdot\hat{r}}}{1-\frac{\hat{r}\cdot\vec{r}'}{r}} \qquad \leftarrow \text{expand } \frac{1}{1-\delta} \sim 1+\delta$$

$$= \frac{\mu_0}{4\pi}\left(\frac{e^{ikr}}{r}\right)\int d^3r' \vec{J}(\vec{r}',\omega)\, e^{-ik\hat{r}\cdot\vec{r}'}\left(1+\frac{\hat{r}\cdot\vec{r}'}{r}\right)$$

when combine with factor $e^{-i\omega t}$ $\quad(\vec{A}(\vec{r},t) = \vec{A}(\vec{r},\omega)e^{-i\omega t})$

this piece gives spherical waves

$$\frac{e^{i(kr-\omega t)}}{r}$$

$\Rightarrow$ oscillating charge <u>radiates</u> outgoing spherical electromagnetic waves

$\int d^3r\, \vec{J}\cdots$ term will determine angular dependence of the radiation

$$\vec{A}(\vec{r},\omega) = \frac{\mu_0}{4\pi}\,\frac{e^{ikr}}{r}\int d^3r'\,\vec{J}(\vec{r}',\omega)\,e^{-ik\hat{r}\cdot\vec{r}'}\left(1+\frac{\hat{r}\cdot\vec{r}'}{r}\right)$$

<u>approximation ②</u> long wave length limit $\lambda \gg d$

$\lambda = \frac{2\pi}{k}$ or $kd \ll 1 \Rightarrow \frac{\omega}{c}d \ll 1$ or $\frac{d}{\tau} \ll c$

$\tau = \frac{2\pi}{\omega}$ Since d is the maximum distance over which charge moves in period of oscilation τ, we see that $kd \ll 1$

$\Rightarrow v \ll c$ where $v \approx \frac{d}{\tau}$ is characteristic velocity with which the charges move

$\Rightarrow \lambda \gg d$ is a <u>non-relativistic</u> approx.

$$kd \ll 1 \Rightarrow e^{-ik\hat{r}\cdot\vec{r}'} \simeq 1 - ik\hat{r}\cdot\vec{r}' + \text{higher order}$$

$$\vec{A}(\vec{r},\omega) = \frac{\mu_0}{4\pi}\,\frac{e^{ikr}}{r}\int d^3r'\,\vec{J}(\vec{r}',\omega)\overbrace{\left[1 + \hat{r}\cdot\vec{r}'\left(\frac{1}{r} - ik\right)\right]}^{(1-ik\hat{r}\cdot\vec{r}')(1+\frac{\hat{r}\cdot\vec{r}'}{r})}$$

+ higher order terms in $\frac{d}{r}$ or kd.

$$\vec{A}(\vec{r},\omega) \equiv \frac{\mu_0}{4\pi}\,\frac{e^{ikr}}{r}\left\{\vec{I}_1 + \left(\frac{1}{r} - ik\right)\vec{I}_2\right\}$$

where $\vec{I}_1 \equiv \int d^3r'\,\vec{J}(\vec{r}',\omega)$

$$\vec{I}_2 \equiv \int d^3r'\,\hat{r}\cdot\vec{r}'\,\vec{J}(\vec{r}',\omega)$$

Now evaluate $\vec{I}_1$ and $\vec{I}_2$

i^{th} component of $\vec{I}_1$

$I_{1i} = \int d^3r' \, j_i(\vec{r}';\omega) = -\int d^3r' \, r_i' \, \vec{\nabla}' \cdot \vec{j}$ via integration by parts
boundary term vanishes as $\vec{j} \to 0$ for finite source

to see this use: trick ① $\vec{\nabla} \cdot (f\vec{g}) - \vec{g} \cdot \vec{\nabla} f = f \vec{\nabla} \cdot \vec{g}$ product rule
apply to right hand side with $f \equiv r_i'$, $\vec{g} \equiv \vec{j}$

$$\Rightarrow -\int d^3r' \, r_i' \vec{\nabla}' \cdot \vec{j} = -\int d^3r' \left\{ \vec{\nabla}' \cdot (r_i' \vec{j}) - \vec{j} \cdot \vec{\nabla}' r_i' \right\}$$

first term: trick ② $\int d^3r' \, \vec{\nabla}' \cdot (r_i' \vec{j}) = \oint_{\text{surface} \to \infty} d\vec{a} \cdot (r_i' \vec{j}) = 0$ as $\vec{j}(r' \to \infty) = 0$ as $\vec{j}$ is localized

second term: trick ③ $\vec{\nabla}' r_i'$ = unit vector in direction i

to see this, consider example: $\vec{\nabla} x = \frac{\partial x}{\partial x} \hat{x} + \frac{\partial x}{\partial y} \hat{y} + \frac{\partial x}{\partial z} \hat{z}$
$= \hat{x} + 0 + 0$

$$\Rightarrow \vec{j} \cdot \vec{\nabla}' r_i' = j_i$$

put together: $-\int d^3r' \, r_i' \vec{\nabla}' \cdot \vec{j} = 0 + \int d^3r' \, j_i = I_{1i}$ as desired

Now use trick ④ charge conservation $\Rightarrow \vec{\nabla}' \cdot \vec{j} = -\frac{\partial \rho}{\partial t} = i\omega \rho(\vec{r}';\omega)$
since $\rho(\vec{r},t) = \rho(\vec{r},\omega) e^{-i\omega t}$

$$\vec{I}_1 = -\int d^3r' \, \vec{r}' \, \vec{\nabla}' \cdot \vec{j} = -i\omega \underbrace{\int d^3r' \, \vec{r}' \rho(\vec{r}';\omega)}_{\vec{p}(\omega)}$$

$\vec{p}(\omega)$ electric dipole moment amplitude at ~~frequency~~ frequency ω

$$\boxed{\vec{I}_1 = -i\omega \vec{p}(\omega)}$$

i^{th} component of $\vec{I}_2$

$$I_{2i} = \int d^3r' \, (\hat{r}\cdot\vec{r}\,')\, \vec{J}_i = \int d^3r' (\hat{r}\cdot\vec{r}\,')\,(\vec{J}\cdot\vec{\nabla}' r_i') \quad \text{by trick ⑤}$$

$$= \sum_{k=1}^{3} \hat{r}_k \int d^3r' \,(r_k' \vec{J})\cdot \vec{\nabla}' r_i' \qquad \text{writing out } \hat{r}\cdot\vec{r}\,' = \sum_{k=1}^{3} \hat{r}_k r_k' \text{ as sum over components}$$

$$= \sum_{k=1}^{3} \hat{r}_k \int d^3r' \left\{ \underbrace{\vec{\nabla}'\cdot(r_i' r_k' \vec{J})}_{=0 \text{ by trick ②}} - r_i' \vec{\nabla}'\cdot(r_k'\vec{J}) \right\} \qquad \text{by trick ① with } \begin{cases} f \equiv r_i' \\ \vec{g} \equiv r_k' \vec{J} \end{cases}$$

$$= -\sum_{k=1}^{3} \hat{r}_k \int d^3r' \left[\underbrace{r_i' r_k' \vec{\nabla}'\cdot\vec{J}}_{\substack{i\omega r_i' r_k' \rho \\ \text{by trick ④}}} + \underbrace{r_i' \vec{J}\cdot\vec{\nabla}' r_k'}_{\substack{r_i' J_k \\ \text{by trick ⑤}}} \right] \qquad \text{expanding } \vec{\nabla}\cdot(r_k'\vec{J}) \text{ as in trick ①}$$

$$= -\sum_{k=1}^{3} \hat{r}_k \int d^3r' \left[r_i' J_k + i\omega r_i' r_k' \rho \right]$$

<u>Last trick:</u> $I_{2i} = \frac{1}{2} I_{2i} + \frac{1}{2} I_{2i}$

$$= \frac{1}{2}\sum_k \hat{r}_k \Big[\underbrace{\int d^3r' \, r_k' J_i}_{\text{from definition of } I_{2i}} - \underbrace{\int d^3r' \left\{ r_i' J_k + i\omega r_i' r_k' \rho \right\}}_{\text{from above manipulations}} \Big]$$

$$\vec{I}_2 = \frac{1}{2}\int d^3r' \Big[\underbrace{(\hat{r}\cdot\vec{r}\,')\vec{J} - (\hat{r}\cdot\vec{J})\vec{r}\,'}_{-\hat{r}\times(\vec{r}\,'\times\vec{J}) \text{ triple product rule}} \Big] - \frac{1}{2}\int d^3r' \, i\omega \, \hat{r}\cdot\vec{r}\,'\,\vec{r}\,' \rho$$

$$= -\frac{1}{2}\hat{r}\times\int d^3r' \,(\vec{r}\,'\times\vec{J}) - \frac{1}{2} i\omega\, \hat{r}\cdot\int d^3r' \,(\vec{r}\,'\vec{r}\,')\rho$$

$$= -\hat{r}\times\vec{m}(\omega) - \frac{1}{2}\frac{i\omega}{3}\,\hat{r}\cdot\overleftrightarrow{Q}'(\omega)$$

where $\vec{m}(\omega) \equiv \frac{1}{2}\int d^3r' \left(\vec{r}' \times \vec{J}(\vec{r}';\omega)\right)$ is magnetic dipole moment

$$\overleftrightarrow{Q}'_{ij}(\omega) = \int d^3r' \; 3\,\vec{r}_i{}'\,\vec{r}_j{}'\,\rho(\vec{r}';\omega)$$

looks very close to electric quadrapole tensor

$$\overleftrightarrow{Q}_{ij} = \int d^3r' \left(3\,\vec{r}_i{}'\,\vec{r}_j{}' - r'^2\delta_{ij}\right)\rho(\vec{r}';\omega)$$

$$\overleftrightarrow{Q}'_{ij} = \overleftrightarrow{Q}_{ij} + \delta_{ij}\int d^3r' \; r'^2\rho(\vec{r}';\omega)$$

$$\vec{I}_2 = -\hat{r}\times\vec{m}(\omega) - \frac{i\omega}{6}\,\hat{r}\cdot\overleftrightarrow{Q}(\omega) - \frac{i\omega}{6}\,\hat{r}\underbrace{\int d^3r' \; r'^2\rho(\vec{r}';\omega)}_{\text{call this } C(\omega) \text{ a scalar}}$$

plug back into $\vec{A}(\vec{r},\omega)$

$$\vec{A}(\vec{r},\omega) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left\{\vec{I}_1 + \left(\frac{1}{r} - ik\right)\vec{I}_2\right\}$$

$$= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left\{\underbrace{-i\omega\vec{p}}_{\text{electric dipole contribution}} - \left(\frac{1}{r} - ik\right)\left(\underbrace{\hat{r}\times\vec{m}}_{\text{magnetic dipole contribution}} + \underbrace{\frac{i\omega}{6}\,\hat{r}\cdot\overleftrightarrow{Q}}_{\text{electric quadrapole contribution}} + \frac{i\omega}{6}\,\hat{r}\,C\right)\right\}$$