where $\vec{m}(\omega) \equiv \frac{1}{2}\int d^3r' \, (\vec{r}' \times \vec{J}(\vec{r}',\omega))$ is magnetic dipole moment

$$\overleftrightarrow{Q}'_{ij}(\omega) = \int d^3r' \; 3\vec{r}_i{}'\vec{r}_j{}' \, \rho(\vec{r}',\omega)$$

looks very close to electric quadrapole tensor

$$\overleftrightarrow{Q}_{ij} = \int d^3r' \left(3\vec{r}_i{}'\vec{r}_j{}' - r'^2\delta_{ij}\right)\rho(\vec{r}',\omega)$$

$$\overleftrightarrow{Q}'_{ij} = \overleftrightarrow{Q}_{ij} + \delta_{ij}\int d^3r' \, r'^2 \rho(\vec{r}',\omega)$$

$$\vec{I}_2 = -\hat{r}\times\vec{m}(\omega) - \frac{i\omega}{6}\hat{r}\cdot\overleftrightarrow{Q}(\omega) - \frac{i\omega}{6}\hat{r}\underbrace{\int d^3r' \, r'^2\rho(\vec{r}',\omega)}_{\text{call this } \mathcal{C}(\omega) \text{ a scalar}}$$

plug back into $\vec{A}(\vec{r},\omega)$

$$\vec{A}(\vec{r},\omega) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left\{\vec{I}_1 + \left(\frac{1}{r} - ik\right)\vec{I}_2\right\}$$

$$= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left\{-i\omega\vec{p} - \left(\frac{1}{r}-ik\right)\left(\hat{r}\times\vec{m} + \frac{i\omega}{6}\hat{r}\cdot\overleftrightarrow{Q} + \frac{i\omega}{6}\hat{r}\,\mathcal{C}\right)\right\}$$

electric dipole contribution — magnetic dipole contribution — electric quadrapole contribution

The last piece which contributes to $\vec{A}$, ie $\frac{i\omega}{6} \overleftrightarrow{Q}\hat{r}\, \frac{e^{ikr}}{r}$ is unimportant - it does not effect the $\vec{E}$ or $\vec{B}$ fields since

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \text{and} \quad \vec{\nabla} \times [f(r)\,\hat{r}] = 0$$

similarly, away from sources, where $\vec{J} = 0$, Ampere's law gives

$$\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A})$$

$$-i\omega \mu_0 \varepsilon_0 \vec{E}(\vec{r}, \omega) = \vec{\nabla} \times (\vec{\nabla} \times \vec{A})$$

since last term doesn't contribute to $\vec{B}$, it doesn't contribute to $\vec{E}$. Formally, we could remove it by making a gauge transformation. Less formally, we will just drop it!

$$\vec{A}(\vec{r}, \omega) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \left\{ -i\omega \vec{p} - \left(\frac{1}{r} - ik\right)\left(\hat{r} \times \vec{m} + \frac{i\omega}{6}\, \hat{r} \cdot \overleftrightarrow{Q}\right) \right\}$$

note: $\left(\frac{1}{r} - ik\right) = -\left(1 + \frac{i}{kr}\right) ik$

lets look at relative strengths of the different terms

far from sources, $\frac{1}{r}$ will be small compared to k, $\frac{1}{r} \ll k \Rightarrow \frac{1}{kr} \ll 1$

radiation zone: just consider those terms in $\vec{A}$ that decrease as slowest powers of $\left(\frac{1}{r}\right)^n$. This will be the $\frac{1}{r}$ terms

Approx ① $d \ll r$

Approx ② $d \ll \lambda$

Radiation Zone ③ $\lambda \ll r$ so $kr \gg 1$

Combine: $d \ll \lambda \ll r$ is RZ

electric dipole term $\vec{p} \simeq q d$ — q is typical charge in source, d is size of source region

magnetic dipole term $\vec{m} = \frac{1}{2}\int d^3r\, \vec{r}\times\vec{j}$ — $\vec{j} \sim q v$ where v is typical velocity

$$\simeq d j \simeq d v q$$

$v \sim \frac{d}{\tau} \sim d\omega \sim dck$

$$\simeq q d^2 \omega \sim q c d^2 k$$

electric quadrapole term $\overleftrightarrow{Q} \sim \int d^3r\, \vec{r}\,\vec{r}\,\rho$

$$\sim q d^2$$

so electric dipole contrib to $\vec{A}$ goes as $\omega \vec{p} \sim q\omega d = qc(kd)$
magnetic dipole contrib to $\vec{A}$ goes as $k\vec{m} \sim q\omega k d^2 = qc(kd)^2$
electric quadrapole contrib to $\vec{A}$ goes as $k\omega \overleftrightarrow{Q} \sim q\omega k d^2 = qc(kd)^2$

Since approx ② assumed (kd) was small
(non relativistic approx: $kd \approx v/c$)
we have an expansion for $\vec{A}$ in powers of (kd)

leading term is the electric dipole term.
next order terms are $\left\{\begin{array}{l}\text{magnetic dipole}\\ \text{electric quadrapole}\end{array}\right\}$ ← these are comparable in strength.

If we kept higher order terms in our expansion, the next terms would be the magnetic quadrapole and electric octopole, both of order $qc(kd)^3$.

Consider now the leading term, the electric dipole term

$$\vec{A}_{E1} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-i\omega)\,\vec{p}(\omega) \qquad \text{"E1"} \equiv \text{electric dipole term}$$

<u>magnetic field</u>

$$\vec{B}_{E1}(\vec{r},\omega) = \vec{\nabla}\times\vec{A}_{E1}(\vec{r},\omega) = \frac{-i\omega\mu_0}{4\pi}\vec{\nabla}\times\left(\frac{e^{ikr}}{r}\vec{p}(\omega)\right)$$

use $\vec{\nabla}\times(f\vec{g}) = (\vec{\nabla}f)\times\vec{g} + f\,\vec{\nabla}\times\vec{g}$ with $f = e^{ikr}$, $\vec{g} = \frac{\vec{p}(\omega)}{r}$

$$\vec{\nabla}e^{ikr} = \hat{r}\frac{\partial}{\partial r}(e^{ikr}) = e^{ikr}\,ik\hat{r} \qquad \text{in spherical coordinates}$$

$$\vec{\nabla}\times\left(\frac{\vec{p}}{r}\right) = \left(\vec{\nabla}\frac{1}{r}\right)\times\vec{p} + \underbrace{\frac{1}{r}\vec{\nabla}\times\vec{p}}_{=0 \text{ since } \vec{p} \text{ is constant}} = -\frac{1}{r^2}\hat{r}\times\vec{p}$$

$$\vec{B}_{E1}(\vec{r},\omega) = \frac{-i\omega\mu_0}{4\pi}\left[e^{ikr}\,ik\frac{\hat{r}\times\vec{p}(\omega)}{r} - e^{ikr}\frac{\hat{r}\times\vec{p}(\omega)}{r^2}\right]$$

use $\omega = ck$

$$\boxed{\vec{B}_{E1} = -\frac{c\mu_0 k^2}{4\pi}\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\vec{p}(\omega)\times\hat{r}}$$

use $\hat{r}\times\vec{p} = -\vec{p}\times\hat{r}$

small compared to 1 when $kr \gg 1$

We define the <u>Radiation Zone</u> limit when $r \gg \lambda \Rightarrow kr \gg 1$
far away on the scale of the wavelength of the radiated wave.
In this limit the 2nd term is small compared to the first
$(1 + i/kr) \approx 1$ and we have

in R.Z. $\boxed{\vec{B}_{E1}(\vec{r},\omega) = -\frac{c\mu_0 k^2}{4\pi}\frac{e^{ikr}}{r}\vec{p}(\omega)\times\hat{r}}$

electric field

from Ampere, $\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \times \vec{B}$ since $\vec{j} = 0$ far from source

$$\Rightarrow \vec{E}_{E1} = \frac{i}{\omega \mu_0 \varepsilon_0} \vec{\nabla} \times \vec{B}_{E1} \quad \text{since } \frac{\partial \vec{E}}{\partial t} = -i\omega \vec{E}$$

$$= \frac{-i}{\omega \mu_0 \varepsilon_0} \frac{c\mu_0}{4\pi} k^2 \vec{\nabla} \times \left(\frac{e^{ikr}}{r} \left(1 + \frac{i}{kr}\right) \vec{p} \times \hat{r} \right)$$

$\omega = ck$

to evaluate the $\vec{\nabla} \times (\;)$ term, use $\vec{\nabla} \times (f\vec{g}) = f \vec{\nabla} \times \vec{g} + \vec{\nabla} f \times \vec{g}$

with $f = e^{ikr}$, $\vec{g} = \frac{1}{r}\left(1 + \frac{i}{kr}\right) \vec{p} \times \hat{r}$

$$\text{then } \vec{\nabla} \times (\;) = \left(\vec{\nabla} e^{ikr}\right) \times \left(\frac{1}{r}\left(1 + \frac{i}{kr}\right) \vec{p} \times \hat{r} \right)$$

$$+ e^{ikr} \vec{\nabla} \times \left(\frac{1}{r}\left(1 + \frac{i}{kr}\right) \vec{p} \times \hat{r} \right)$$

But in the radiation zone we can ignore the second term, since

$$\vec{\nabla} \times \left(\frac{1}{r}\left(1 + \frac{i}{kr}\right) \vec{p} \times \hat{r} \right) \sim \frac{1}{r^2}$$

we see this by noting that

$$\vec{\nabla} \frac{1}{r} \sim \frac{1}{r^2}, \quad \vec{\nabla} \frac{1}{r^2} \sim \frac{1}{r^3}$$

$$\vec{\nabla} \hat{r} \sim \frac{\partial}{\partial x} \frac{\vec{r}}{|\vec{r}|} = \frac{\hat{x}}{|\vec{r}|} - \frac{\vec{r}}{|\vec{r}|^2} \frac{x}{|\vec{r}|} = \frac{\hat{x}}{|\vec{r}|} - \frac{x\hat{r}}{|\vec{r}|^2} \sim O\left(\frac{1}{r}\right)$$

So keep only 1st term in Radiation Zone and we get

$$\vec{E}_{E1} = \frac{-i}{\omega \mu_0 \varepsilon_0} \frac{c\mu_0}{4\pi} k^2 \underbrace{\left(ike^{ikr} \hat{r}\right)}_{= \vec{\nabla} e^{ikr}} \times \left(\frac{1}{r}\left(1 + \frac{i}{kr}\right) \vec{p} \times \hat{r} \right)$$

ignore in RZ

use $\omega = ck$

$$\boxed{\vec{E}_{E1} = \frac{k^2}{4\pi \varepsilon_0} \frac{e^{ikr}}{r} \hat{r} \times (\vec{p} \times \hat{r})}$$

<u>electric field</u> - full calculation without making Radiation Zone approximation

from Ampere, $\mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \vec{\nabla}\times\vec{B}$, since $\vec{J}=0$ far from source

$$\Rightarrow \quad \vec{E}_{E1} = \frac{i}{\omega\mu_0\varepsilon_0}\,\vec{\nabla}\times\vec{B}_{E1} \qquad \text{since } \frac{\partial\vec{E}}{\partial t} = -i\omega\vec{E}$$

$$= \frac{-i}{\omega\mu_0\varepsilon_0}\,\frac{c\mu_0}{4\pi}k^2\,\vec{\nabla}\times\left(\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\vec{p}\times\hat{r}\right)$$

$\omega = ck$

to evaluate $\vec{\nabla}\times(\ )$, use $\vec{\nabla}\times(f\vec{g}) = f\,\vec{\nabla}\times\vec{g} + \vec{\nabla}f\times\vec{g}$

with $f = \frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)$ and $\vec{g} = \vec{p}\times\hat{r}$

$$\vec{\nabla}\times(\ ) = \frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\vec{\nabla}\times(\vec{p}\times\hat{r}) + \vec{\nabla}\left(\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\right)\times(\vec{p}\times\hat{r})$$

evaluate second term: $\vec{\nabla}\left(\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\right) = \frac{\partial}{\partial r}\left(\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\right)\hat{r}$ in spherical coords

$$= e^{ikr}\left[ik\left(\frac{1}{r}+\frac{i}{kr^2}\right) - \frac{1}{r^2} - \frac{2i}{kr^3}\right]\hat{r}$$

$$= \frac{e^{ikr}}{r}\left[ik - \frac{2}{r} - \frac{2i}{kr^2}\right]\hat{r}$$

evaluate first term: $\vec{\nabla}\times(\vec{p}\times\hat{r}) = \vec{p}\,(\vec{\nabla}\cdot\hat{r}) - (\vec{p}\cdot\vec{\nabla})\hat{r}$ cover Griffiths

where $\vec{\nabla}\cdot\hat{r} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2) = \frac{2}{r}$ evaluating in spherical coordinates

and $(\vec{p}\cdot\vec{\nabla})\hat{r} = \sum_k p_k \frac{\partial\hat{r}}{\partial r_k}$

unit vector in k direction

$$\text{where } \frac{\partial\hat{r}}{\partial r_k} = \frac{\partial}{\partial r_k}\left(\frac{\vec{r}}{r}\right) = \vec{r}\left(-\frac{1}{r^2}\frac{\partial r}{\partial r_k}\right) + \frac{\hat{e}_k}{r}$$

$$= \vec{r}\left(\frac{-1}{r^2}\frac{r_k}{r}\right) + \frac{\hat{e}_k}{r} \qquad \text{as } \frac{\partial r}{\partial r_k} = \frac{r_k}{r}$$

so $\quad \vec{\nabla}\times(\vec{p}\times\hat{r}) = \dfrac{2\vec{p}}{r} - \sum_k p_k\left(\dfrac{-\vec{r}\, r_k}{r^3} + \dfrac{\hat{e}_k}{r}\right)$

$$= \frac{2\vec{p}}{r} + \frac{\vec{r}\;\vec{p}\cdot\vec{r}}{r^3} - \frac{\vec{p}}{r}$$

$$= \underline{\frac{\vec{p} + \hat{r}(\vec{p}\cdot\hat{r})}{r}} \qquad \text{using } \hat{r} = \frac{\vec{r}}{r}$$

putting all the pieces together

$$\vec{E}_{E1} = \frac{-ik}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\Bigg[\left(1+\frac{i}{kr}\right)\frac{\vec{p}+\hat{r}(\vec{p}\cdot\hat{r})}{r} + \left(ik - \frac{2}{r} - \frac{2i}{kr^2}\right)\cdot\underbrace{\hat{r}\times(\vec{p}\times\hat{r})}_{\vec{p}-\hat{r}(\vec{p}\cdot\hat{r})}\Bigg]$$

order by powers of $\frac{1}{r}$

$$= -\frac{ik}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\Bigg[ik\left(\vec{p}-\hat{r}(\vec{p}\cdot\hat{r})\right) + \frac{1}{r}\left(1+\frac{i}{kr}\right)\left(\vec{p}+\hat{r}(\vec{p}\cdot\hat{r})\right) - \frac{2}{r}\left(1+\frac{i}{kr}\right)\left(\vec{p}-\hat{r}(\vec{p}\cdot\hat{r})\right)\Bigg]$$

$$= \frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\Bigg[\vec{p}-\hat{r}(\vec{p}\cdot\hat{r}) - \frac{i}{kr}\left(1+\frac{i}{kr}\right)\left(\vec{p}+\hat{r}(\vec{p}\cdot\hat{r}) - 2\vec{p} + 2\hat{r}(\vec{p}\cdot\hat{r})\right)\Bigg]$$

$$\boxed{\vec{E}_{E1} = \frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left[\vec{p}-\hat{r}(\vec{p}\cdot\hat{r}) - \frac{i}{kr}\left(1+\frac{i}{kr}\right)\left(3\hat{r}(\vec{p}\cdot\hat{r})-\vec{p}\right)\right]}$$

<u>radiation zone approx</u> $kr \gg 1$ keep only terms of order $\frac{1}{r}$

$$\boxed{\vec{E}_{E1} = \frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\underbrace{\left[\vec{p}-\hat{r}(\vec{p}\cdot\hat{r})\right]}_{\hat{r}\times(\vec{p}\times\hat{r})}}$$

<u>Radiation zone limit</u>

$$\left.\begin{aligned}\vec{E}_{E1} &= \frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\,\hat{r}\times(\vec{p}\times\hat{r}) \\ \vec{B}_{E1} &= -\frac{c\mu_0}{4\pi}k^2\frac{e^{ikr}}{r}\,\vec{p}\times\hat{r}\end{aligned}\right\}$$

radiation zone fields in electric dipole approx

$\vec{E}$ and $\vec{B}$ are outwards traveling spherical waves $\sim \frac{e^{ikr}}{r}$

using $\mu_0\varepsilon_0 = 1/c^2$

$$\frac{|\vec{B}_{E1}|}{|\vec{E}_{E1}|} = \frac{c\mu_0}{4\pi}\cdot 4\pi\varepsilon_0 = c\mu_0\varepsilon_0 = \frac{c}{c^2} = \frac{1}{c}$$

just as for plane waves in vacuum

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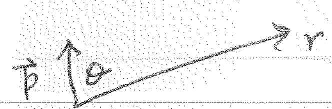
if choose coordinates so that $\vec{p}$ is along $\hat{z}$ axis



$\vec{p}\times\hat{r} = \hat{\phi}\sin\theta\, p$

$\hat{r}\times(\vec{p}\times\hat{r}) = p\sin\theta(\hat{r}\times\hat{\phi})$

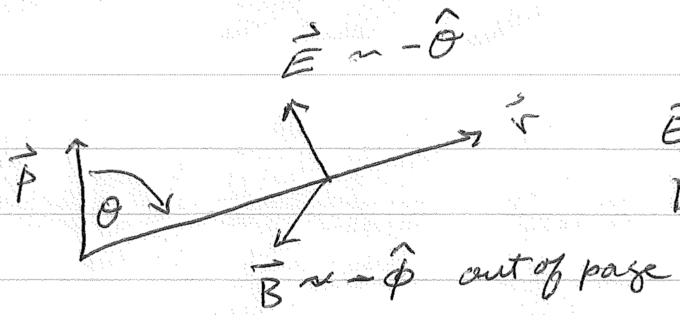
$= -\hat{\theta}\sin\theta\, p$

$$\vec{E}_{E1} = \frac{-k^2 p}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\sin\theta\,\hat{\theta}$$

$$\vec{B}_{E1} = -\frac{c\mu_0}{4\pi}k^2 p\,\frac{e^{ikr}}{r}\sin\theta\,\hat{\phi}$$

$\hat{\theta}$ & $\hat{\phi}$ are spherical coord basis vectors

Note: Above assumes that $\vec{p}$ is a real valued vector. In general it is possible $\vec{p}$ may be complex, $\vec{p} = \vec{p}_1 + i\vec{p}_2$, and that $\vec{p}_1$ and $\vec{p}_2$ may point in different directions!



$\vec{E}$ is in the plane containing $\vec{p}$ and $\hat{r}$

$\vec{B}$ is $\perp$ to this plane

$\vec{E}_{E1}$ and $\vec{B}_{E1}$ are orthogonal, as in a plane wave,
and both are orthogonal to the direction of propagation $\hat{r}$

$\Rightarrow$ oscillating source emits spherical electromagnetic waves

What is Power emitted?

Poynting vector: $\vec{S}_{E1}(\vec{r},t) = \frac{1}{\mu_0} \mathrm{Re}\left[\vec{E}_{E1}(\vec{r},t)\right] \times \mathrm{Re}\left[\vec{B}_{E1}(\vec{r},t)\right]$

$$\mathrm{Re}\left[\vec{E}_{E1}(\vec{r},t)\right] = \mathrm{Re}\left[-\frac{k^2 p}{4\pi\varepsilon_0} \frac{e^{ikr}}{r} \sin\theta\, \hat{\theta}\, e^{-i\omega t}\right]$$

$$= \frac{-k^2 p}{4\pi\varepsilon_0} \frac{\cos(kr-\omega t)}{r} \sin\theta\, \hat{\theta}$$

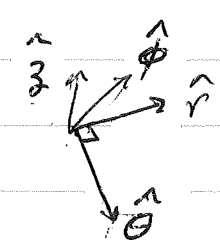
$$\mathrm{Re}\left[\vec{B}_{E1}(\vec{r},t)\right] = \mathrm{Re}\left[-\frac{c\mu_0}{4\pi} k^2 p \frac{e^{ikr}}{r} \sin\theta\, \hat{\varphi}\, e^{-i\omega t}\right]$$

$$= -\frac{c\mu_0 k^2 p}{4\pi} \frac{\cos(kr-\omega t)}{r} \sin\theta\, \hat{\varphi}$$

*** Assuming $\vec{p}$ is a real valued vector ***

$$\vec{S}_{E1} = \frac{1}{\mu_0} \frac{k^2 p}{4\pi\varepsilon_0} \frac{c\mu_0 k^2 p}{4\pi} \frac{\cos^2(kr-\omega t)}{r^2} \sin^2\theta\, (\hat{\theta} \times \hat{\varphi})$$

$$= \frac{c\, k^4 p^2}{(4\pi)^2 \varepsilon_0} \frac{\cos^2(kr-\omega t)}{r^2} \sin^2\theta\, \hat{r}$$



Average over one period of oscillation $\langle \cos^2(kr-\omega t)\rangle = \frac{1}{2}$

$$\langle \vec{S}_{E1} \rangle = \frac{ck^4 p^2}{2(4\pi)^2 \varepsilon_0} \frac{\sin^2\theta}{r^2} \hat{r}$$

Note, the $\frac{1}{r^2}$ is important for energy conservation.
If we integrate $\langle \vec{S}_{E1} \rangle \cdot \hat{r}$ over the surface of a sphere of radius r, the result is independent of r.

average energy flux flowing through an element of area at spherical angles θ, ϕ is

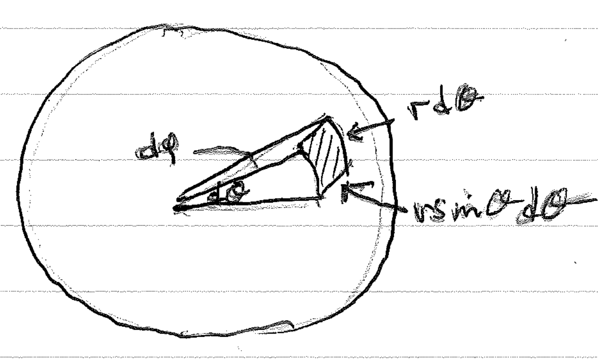
$$\text{power } dP_{E1} = \hat{r} \cdot \langle \vec{S}_{E1} \rangle \underbrace{r^2 \sin\theta \, d\theta \, d\phi}$$

differential area on surface of sphere spanned by $d\theta$ and $d\phi$
$= r^2 d\Omega$

$d\Omega = \sin\theta \, d\theta \, d\phi$ differential solid angle

$$dP_{E1} = \hat{r} \cdot \langle \vec{S}_{E1} \rangle r^2 d\Omega$$

$$\frac{dP_{E1}}{d\Omega} = \hat{r} \cdot \langle \vec{S}_{E1} \rangle r^2 = \frac{ck^4 p^2}{2(4\pi)^2 \varepsilon_0} \sin^2\theta$$



$da = r^2 \sin\theta \, d\theta \, d\phi$
$= r^2 d\Omega$