Potentials from a moving point charge

Liénard - Wiechert potentials

In general

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \int dt' \, \frac{\rho(\vec{r}',t')}{|\vec{r}-\vec{r}'|} \, \delta\!\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3r' \int dt' \, \frac{\vec{j}(\vec{r}',t')}{|\vec{r}-\vec{r}'|} \, \delta\!\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)$$

For a pt charge moving on trajectory $\vec{r}_0(t)$ with velocity $\vec{v}(t) = \frac{d\vec{r}_0}{dt}$

$$\rho(\vec{r},t) = q\,\delta(\vec{r}-\vec{r}_0(t))$$

$$\vec{j}(\vec{r},t) = q\,\vec{v}(t)\,\delta(\vec{r}-\vec{r}_0(t))$$

Substitute in

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \int dt' \, \frac{q\,\delta(\vec{r}'-\vec{r}_0(t'))}{|\vec{r}-\vec{r}'|} \, \delta\!\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)$$

do $\int d^3r'$ using $\delta(\vec{r}'-\vec{r}_0(t'))$ then

$$= \frac{1}{4\pi\varepsilon_0}\int dt' \, \frac{q}{|\vec{r}-\vec{r}_0(t')|} \, \delta\!\left(t-t'-\frac{|\vec{r}-\vec{r}_0(t')|}{c}\right)$$

now we want to do $\int dt'$ using the $\delta\!\left(t-t'-\frac{|\vec{r}-\vec{r}_0(t')|}{c}\right)$

the δ-function is of the form $\delta(g(t'))$ where

$$g(t') = t' - t + \frac{|\vec{r} - \vec{r}_0(t')|}{c}$$

to evaluate, change variable of integration from t' to g

$$\int_{-\infty}^{\infty} dt'\, f(t')\, \delta(g(t')) = \int_{g(-\infty)}^{g(\infty)} f(t')\, \delta(g)\left(\frac{dt'}{dg}\right) dg$$

$$= f(t')\,\frac{dt'}{dg} = \frac{f(t')}{\left(\frac{dg}{dt'}\right)}$$ evaluated at t' such that $g(t') = 0$

Here: $f(t') = \dfrac{q}{|\vec{r} - \vec{r}_0(t')|}$

$$t' = t - \frac{|\vec{r} - \vec{r}_0(t')|}{c}$$ is the retarded time

$$\frac{dg}{dt'} = 1 + \frac{1}{c}\frac{d}{dt'}|\vec{r} - \vec{r}_0(t')|$$

$$\frac{d}{dt}\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} = \frac{-\left[(x-x_0)\frac{dx_0}{dt} + (y-y_0)\frac{dy_0}{dt} + (z-z_0)\frac{dz_0}{dt}\right]}{\sqrt{\quad}}$$

$$= \frac{-(\vec{r} - \vec{r}_0(t'))\cdot\vec{v}(t')}{|\vec{r} - \vec{r}_0(t')|} \equiv -\hat{n}(t')\cdot\vec{v}(t')$$

where $\hat{n}(t') \equiv \dfrac{\vec{r} - \vec{r}_0(t')}{|\vec{r} - \vec{r}_0(t')|}$

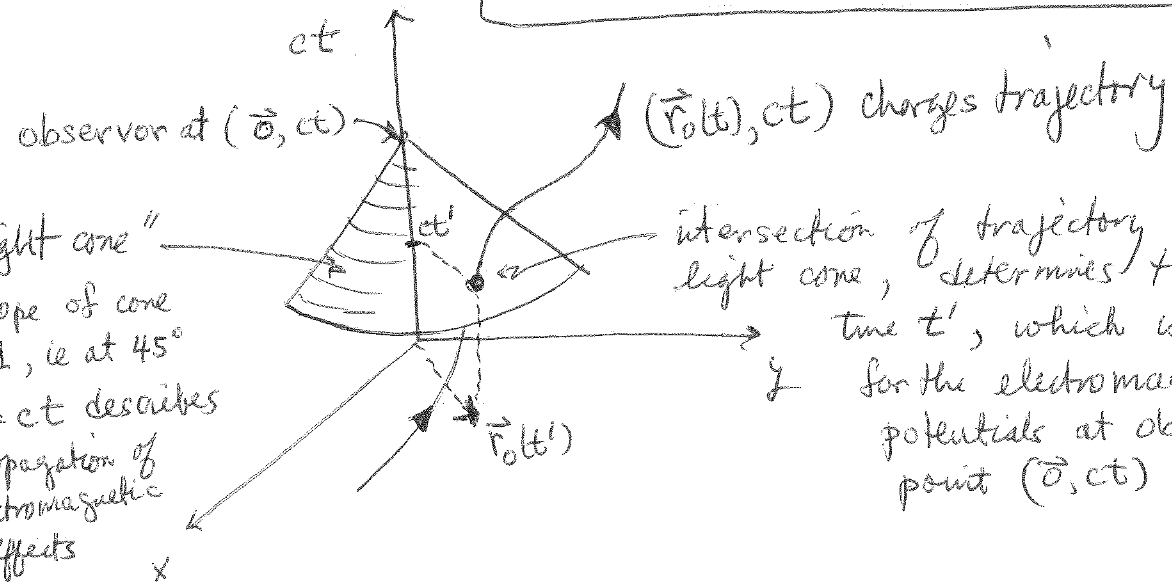
unit vector from charges position, at retarded time t', to observer at position r, time t.

$$\boxed{\begin{aligned} V(\vec{r},t) &= \frac{q}{4\pi\varepsilon_0\,|\vec{r}-\vec{r}_0(t')|}\;\frac{1}{1-\frac{1}{c}\,\hat{n}(t')\cdot\vec{v}(t')} \\ \text{similarly:}\quad \vec{A}(\vec{r},t) &= \frac{\mu_0}{4\pi}\;\frac{q\,\vec{v}(t')}{|\vec{r}-\vec{r}_0(t')|}\;\frac{1}{1-\frac{1}{c}\,\hat{n}(t')\cdot\vec{v}(t')} \\ t' &= t-\frac{|\vec{r}-\vec{r}_0(t')|}{c} \end{aligned}}$$

Liénard - Wiechert potentials

using $\mu_0 = \frac{1}{\varepsilon_0 c^2}$ $\Longrightarrow$ $$\boxed{\vec{A}(\vec{r},t) = \frac{\vec{v}(t')}{c^2}\,V(\vec{r},t)}$$

graphically:



"light cone"
slope of cone is 1, ie at 45°
r=ct describes propagation of electromagnetic effects

intersection of trajectory with light cone, determines the retarded time t', which is responsible for the electromagnetic potentials at observation point $(\vec{0}, ct)$

what is observed at the apex of the cone, at point $(\vec{0}, ct)$ can only be caused by events which occurred somewhere on the surface of the light ~~[illegible]~~ cone.

pts r' on light cone determined by eqn $ct' = ct - r'$ as observer at $r=0$

$\Rightarrow r' = ct - ct'$

points inside the light cone arrive at the observer at times earlier than t
points outside the light cone arrive at the observer at times later than t

Potential from charge moving with constant velocity

$\vec{r}_0(t) = \vec{v}\, t$

find retarded time $t' = t - \frac{|\vec{r} - \vec{r}_0(t')|}{c} \Rightarrow c(t-t') = |\vec{r} - \vec{r}_0(t')|$

$\longrightarrow c^2(t-t')^2 = |\vec{r} - \vec{r}_0(t')|^2 = r^2 + v^2 t'^2 - 2\vec{r}\cdot\vec{v}\, t'$

$c^2t^2 + c^2t'^2 - 2c^2 t t' = r^2 + v^2 t'^2 - 2\vec{r}\cdot\vec{v}\, t'$

quadratic equation in t'

$\longrightarrow (c^2 - v^2)t'^2 - 2(c^2 t - \vec{r}\cdot\vec{v})t' + c^2t^2 - r^2 = 0$

quadratic formula

$$\boxed{t' = \frac{(c^2 t - \vec{r}\cdot\vec{v}) \pm \sqrt{(c^2 t - \vec{r}\cdot\vec{v})^2 - (c^2t^2 - r^2)(c^2 - v^2)}}{(c^2 - v^2)} \quad (*)}$$

to get sign correct, note that when $v = 0$, we want $t' = t - \frac{r}{c}$. Apply to above:

$$t' = \frac{c^2 t \pm \sqrt{c^4t^2 - c^4t^2 + c^2r^2}}{c^2}$$

$$= t \pm \frac{r}{c} \Rightarrow \underline{\text{take } \ominus \text{ sign}}$$

scalar potential $V(\vec{r},t) = \frac{q}{4\pi\varepsilon_0} \frac{1}{|r - r_0(t')|\left(1 - \frac{1}{c}\hat{n}(t')\cdot\vec{v}(t')\right)}$

here $\vec{v}(t') = \vec{v}$ constant $\qquad \hat{n}(t') = \frac{\vec{r} - \vec{r}_0(t')}{|\vec{r} - \vec{r}_0(t')|}$

look at denominator:

$$\underbrace{|\vec{r} - \vec{r}_0(t')|}_{= c(t-t')}\left(1 - \frac{1}{c}\hat{n}(t')\cdot\vec{v}\right) = \underbrace{|\vec{r} - \vec{v}t'|}_{= c(t-t')} - \frac{1}{c}(\vec{r} - \vec{v}t')\cdot\vec{v}$$

$$= c(t-t') - \frac{1}{c}\vec{r}\cdot\vec{v} + \frac{v^2 t'}{c} = ct - \frac{\vec{r}\cdot\vec{v}}{c} - c\left(1 - \frac{v^2}{c^2}\right)t'$$

$$= \frac{1}{c}\left[\, c^2 t - \vec{r}\cdot\vec{v} - (c^2 - v^2)\, t' \,\right]$$ insert (*) for t'

$$= \frac{1}{c}\left[\underbrace{c^2 t - \vec{r}\cdot\vec{v} - (c^2 t - \vec{r}\cdot\vec{v})}_{\text{cancel}} + \sqrt{(c^2 t^2 - \vec{r}\cdot\vec{v})^2 - (c^2 t - r^2)(c^2 - v^2)}\,\right]$$

$$= \frac{1}{c}\sqrt{(c^2 t - \vec{r}\cdot\vec{v})^2 - (c^2 t^2 - r^2)(c^2 - v^2)}$$

simplify the square root

$$= \frac{1}{c}\sqrt{\cancel{c^4 t^2} + (\vec{r}\cdot\vec{v})^2 - 2c^2 t\, \vec{r}\cdot\vec{v} - \cancel{c^4 t^2} + r^2 c^2 + c^2 v^2 t^2 - r^2 v^2}$$

$$= \frac{1}{c}\sqrt{c^2(\vec{r} - \vec{v}t)^2 + (\vec{r}\cdot\vec{v})^2 - r^2 v^2}$$

$$= \sqrt{(\vec{r} - \vec{v}t)^2 + \frac{(\vec{r}\cdot\vec{v})^2}{c^2} - \frac{r^2 v^2}{c^2}}$$

$$\boxed{V(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{(\vec{r} - \vec{v}t)^2 + \frac{(\vec{r}\cdot\vec{v})^2}{c^2} - \frac{r^2 v^2}{c^2}}}}$$

Note, for $\left(\frac{v}{c}\right)^2 \ll 1$

$$V(\vec{r},t) \approx \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{(\vec{r} - \vec{v}t)^2}} = \frac{q}{4\pi\varepsilon_0}\frac{1}{|\vec{r} - \vec{r}_0(t)|^2}$$

"quasistatic" limit
looks like instantaneous
Coulomb potential

vector potential

$$\vec{A}(\vec{r},t) = \frac{\vec{v}}{c^2} V(\vec{r},t)$$

fields from charges moving with constant $\vec{v}$

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}V - \frac{\vec{v}}{c^2}\frac{\partial V}{\partial t} \quad \text{as} \quad \vec{A} = \frac{\vec{v}}{c^2}V$$

$$-\vec{\nabla}V = \frac{q}{4\pi\varepsilon_0}\frac{(\frac{1}{2})}{(\ )^{3/2}}\left[\vec{\nabla}(\vec{r}-\vec{v}t)^2 + \frac{1}{c^2}\vec{\nabla}(\vec{r}\cdot\vec{v})^2 - \frac{v^2}{c^2}\vec{\nabla}r^2\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{(\ )^{3/2}}\left[\vec{r}-\vec{v}t + \frac{\vec{v}}{c^2}(\vec{r}\cdot\vec{v}) - \frac{v^2}{c^2}\vec{r}\right]$$

$$-\frac{\partial V}{\partial t} = \frac{q}{4\pi\varepsilon_0}\frac{(\frac{1}{2})}{(\ )^{3/2}}\left[\frac{\partial}{\partial t}(\vec{r}-\vec{v}t)^2\right] = \frac{q}{4\pi\varepsilon_0}\frac{(-1)}{(\ )^{3/2}}\left[(\vec{r}-\vec{v}t)\cdot\vec{v}\right]$$

$$\vec{E} = -\vec{\nabla}V - \frac{\vec{v}}{c^2}\frac{\partial V}{\partial t} = \frac{q}{4\pi\varepsilon_0}\frac{1}{(\ )^{3/2}}\Big[\vec{r}-\vec{v}t + \cancel{\frac{\vec{v}}{c^2}(\vec{r}\cdot\vec{v})} - \frac{v^2}{c^2}\vec{r}$$
$$- \frac{\vec{v}}{c^2}\left(\cancel{\vec{r}\cdot\vec{v}} - v^2 t\right)\Big]$$

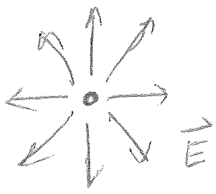
$$\boxed{\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{(\vec{r}-\vec{v}t)(1-v^2/c^2)}{\left((\vec{r}-\vec{v}t)^2 + (\frac{\vec{r}\cdot\vec{v}}{c})^2 - (\frac{rv}{c})^2\right)^{3/2}}}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times\left(\frac{\vec{v}}{c^2}V\right) = (\vec{\nabla}V)\times\frac{\vec{v}}{c^2} = \left(-\vec{E} - \frac{\partial\vec{A}}{\partial t}\right)\times\frac{\vec{v}}{c^2}$$

$$= -\vec{E}\times\frac{\vec{v}}{c^2} - \frac{\partial}{\partial t}\left(\frac{\vec{v}}{c^2}V\right)\times\frac{\vec{v}}{c^2} = \frac{\vec{v}}{c^2}\times\vec{E} - \frac{\partial V}{\partial t}\underbrace{\frac{\vec{v}}{c^2}\times\frac{\vec{v}}{c^2}}_{=0}$$

$$\boxed{\vec{B}(\vec{r},t) = \frac{\vec{v}}{c^2}\times\vec{E}(\vec{r},t)}$$

Note: The factor $1-v^2/c^2$
reminds one of
special relativity!



for $v^2 \ll c^2$

$$\vec E(\vec r,t) \simeq \frac{q}{4\pi\varepsilon_0}\frac{(\vec r-\vec v t)}{|\vec r-\vec v t|^3}$$ instantaneous Coulomb field

$$\vec B(\vec r,t) = \frac{\vec v}{c^2}\times\vec E = \frac{1}{4\pi\varepsilon_0 c^2}\, q\frac{\vec v\times(\vec r-\vec v t)}{|\vec r-\vec v t|^3}$$

$$= \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\vec j(\vec r\,')\times(\vec r-\vec r\,')}{|\vec r-\vec r\,'|^3}$$

with $\vec j(\vec r\,') = q\vec v\,\delta(\vec r\,'-\vec v t)$

so $\vec B$ looks just like Biot-Savart law, even though for moving charge, $\vec\nabla\cdot\vec j \neq 0$.

Since a charge moving with constant $\vec v$ can always be considered to be at rest in another ~~E~~ frame of reference, the formulas above for $\vec E$ and $\vec B$ for charge moving with $\vec v$, will tell us how $\vec E$ and $\vec B$ transform under a Lorentz transformation between two frames in relative uniform motion. ~~Thus~~ In fact, the Lorentz transf so defined, was first discovered as a law of electrodynamics, before the theory of special relativity was discovered.

For a pt charge in state of arbitrary motion ($\dot{\vec{v}} \neq 0$)
one finds (see text)

$$\boxed{\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\,\frac{|\vec{r}-\vec{r}_0(t')|}{\left([\vec{r}-\vec{r}_0(t')]\cdot[c\hat{n}(t')-\vec{v}(t')]\right)^3}\Big[[c\hat{n}(t')-\vec{v}(t')](c^2-v^2) + (\vec{r}-\vec{r}_0(t'))\times\left([c\hat{n}(t')-\vec{v}(t')]\times\vec{a}(t')\right)\Big]}$$

where $t' = t - \frac{|\vec{r}-\vec{r}_0(t')|}{c}$ is retarded time

$$\hat{n}(t') = \frac{\vec{r}-\vec{r}_0(t')}{|\vec{r}-\vec{r}_0(t')|} \qquad \vec{v} = \frac{d\vec{r}_0}{dt}, \quad \vec{a} = \frac{d^2\vec{r}_0}{dt^2}$$

First term $\sim \frac{1}{r^2}$ is "generalized Coulomb field" or "velocity field"

second term $\sim \frac{1}{r}$ is "acceleration field"

$$\boxed{\vec{B}(\vec{r},t) = \frac{\hat{n}(t')}{c}\times\vec{E}(\vec{r},t)}$$

$\vec{B}$ always $\perp \vec{E}$, and $\vec{B} \perp \hat{n}$, vector from observer at $\vec{r}$ to retarded point $\vec{r}_0(t')$.

Can use above to write down total force between two point charges, in arbitrary states of motion (see text)

Can use to construct $\vec{S}$ and get power radiated by an accelerating point charge - in non-relativistic limit, one recovers Larmor's formula - see text