Some Comments on quadrupoles

$$\overleftrightarrow{Q} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Q_{zz} \end{pmatrix}$$

shouldn't $\overleftrightarrow{Q}$ be traceless?

proper definition was $\overleftrightarrow{Q} = \int d^3r \left(3\vec{r}\,\vec{r} - r^2 \overleftrightarrow{I}\right) \rho(\vec{r})$

This $\overleftrightarrow{Q}$ is traceless $\sum_{i=1}^{3} Q_{ii} = 0$

But we saw that we could just as well use

$$\overleftrightarrow{Q}' = \int d^3r \; 3\vec{r}\,\vec{r} \; \rho(\vec{r})$$

which is not traceless. so it is really $\overleftrightarrow{Q}'_{ij} = Q_{zz}\,\delta_{iz}\delta_{jz}$

If we want a $\overleftrightarrow{Q}$ with the same physical effects it would be

$$\overleftrightarrow{Q} = \overleftrightarrow{Q}' - \left[\tfrac{1}{3} \operatorname{Trace} \overleftrightarrow{Q}'\right] \overleftrightarrow{I} = \begin{pmatrix} -\frac{1}{3}Q_{zz} & 0 & 0 \\ 0 & -\frac{1}{3}Q_{zz} & 0 \\ 0 & 0 & \frac{2}{3}Q_{zz} \end{pmatrix}$$

To evaluate $\vec{S}$ we need terms like $(\hat{r}\cdot\overleftrightarrow{Q})^2$ and $(\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r})^2$

This might lead one to think that we should express $\overleftrightarrow{Q}$ in spherical coordinates, ie

$$\begin{pmatrix} Q_{rr} & Q_{r\theta} & Q_{r\phi} \\ Q_{r\theta} & Q_{\theta\theta} & Q_{\theta\phi} \\ Q_{r\phi} & Q_{r\theta} & Q_{\phi\phi} \end{pmatrix}$$

can we do this? is it worthwhile?

Consider $Q_{r\theta} = \hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{\theta}$

$$= \hat{r}\cdot\left[\int d^3r'\, 3\vec{r}'\vec{r}'\rho\right]\cdot\hat{\theta} = \int d^3r'\, 3(\hat{r}\cdot\vec{r}')(\vec{r}'\cdot\hat{\theta})\rho$$

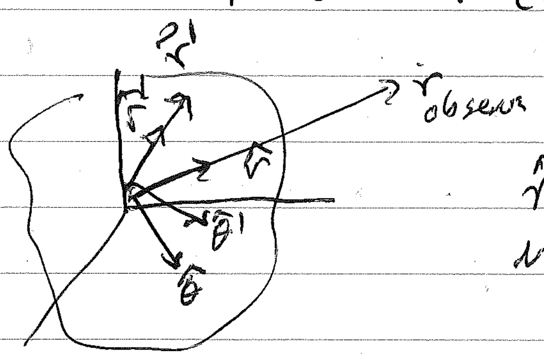
can one say $\hat{r}\cdot\vec{r}' = r'$ and $\vec{r}'\cdot\hat{\theta} = 0$?

NO! because $\vec{r}'$ is the integration variable and $\hat{r}$, $\hat{\theta}$ are fixed unit vectors defined by the position of the observer

ie $\vec{r}' = r'\hat{r}'$

$$\hat{r}\cdot\vec{r}' = r'(\hat{r}\cdot\hat{r}')$$
$$\vec{r}'\cdot\hat{\theta} = r'\underbrace{(\hat{\theta}\cdot\hat{r}')}_{\neq 0}$$

Doing it this way, the components of $\overleftrightarrow{Q}$ would depend on the direction $\hat{r}$ to the observer. Not good!



$\hat{r}'$ varies as integration variable changes.

Better to do calc in Cartesian coordinates using

$$\overleftrightarrow{Q} = \begin{pmatrix} Q_{xx} & Q_{xy} & Q_{xz} \\ Q_{xy} & Q_{yy} & Q_{yz} \\ Q_{xz} & Q_{xy} & Q_{zz} \end{pmatrix}$$ and $\hat{r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

Inertial frames of reference: Set of frames of reference which move at constant velocity with respect to each other

Special Relativity

1) Speed of light is constant in all inertial frames of reference
2) Physical laws must look the same in all inertial frames of reference - there is no experiment that can determine the "absolute" velocity of any inertial frame

$\Rightarrow$ If a flash of light goes off at the origin of some coord system, the outgoing wavefronts look spherical in all inertial frames.

Equation of wavefront is $r^2 - c^2t^2 = 0$

$\Rightarrow$ (x, y, z, t) coords in one inertial frame K

(x', y', z', t') coords in another inertial frame K' that moves with velocity $\vec{v} = v\hat{x}$ with respect to K.

What is the transformation that relates coords in K' to coords in K

$y = y'$, $z = z'$ (origins of K and K' coincide when $t = t' = 0$)

$\Rightarrow$ $c^2t^2 - x^2 = c^2t'^2 - x'^2$ ← equation of wave front as seen by K and as seen by K'

$$\Rightarrow \quad \frac{(ct+x)}{(ct'+x')}\frac{(ct-x)}{(ct'-x')} = 1$$

Expect transformation to be linear

(if transf was not linear, a particle moving at constant $\vec{v}$ in one frame might look accelerated in another frame)

$$\Rightarrow \quad ct' + x' = (ct+x)f$$

$$ct' - x' = (ct-x)f^{-1}$$

for some constant f write $f = e^{-y}$

y is called the "rapidity"

Solve for ct' and x' in terms of ct and x

$$ct' = ct\left(\frac{e^y + e^{-y}}{2}\right) - x\left(\frac{e^y - e^{-y}}{2}\right)$$

$$x' = -ct\left(\frac{e^y - e^{-y}}{2}\right) + x\left(\frac{e^y + e^{-y}}{2}\right)$$

$$ct' = ct\cosh y - x\sinh y$$
$$x' = -ct\sinh y + x\cosh y$$

<u>meaning of parameter y</u>

the origin of K (at $x=0$) has trajectory $x' = -vt'$ in K'

$$\Rightarrow \quad \frac{x'}{t'} = -v$$

from transformation above, with $x=0$, we get

$$\frac{x'}{ct'} = \frac{-ct\sinh y}{ct\cosh y} = -\tanh y$$

So $\quad \frac{v}{c} = \tanh y$

$$\Rightarrow \quad \cosh y = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \equiv \gamma$$

$$\sinh y = \left(\frac{v}{c}\right)\gamma$$

<u>Lorentz Transformation</u>
$$\begin{cases} ct' = \gamma ct - \gamma\left(\frac{v}{c}\right)x \\ x' = -\gamma\left(\frac{v}{c}\right)ct + \gamma x \end{cases}$$

Inverse Lorentz transform obtained by taking $v \to -v$ in above

$$\begin{cases} ct = \gamma ct' + \gamma\left(\frac{v}{c}\right)x' \\ x = \gamma\left(\frac{v}{c}\right)ct' + \gamma x' \end{cases}$$

Time Dilation

Consider a clock located at the origin in frame K' that moves with velocity $v\hat{x}$ as seen from "lab" frame K.

The clock in frame K' ticks at $t_1' = 0$ and $t_2' = T_0$

Time between ticks in frame K' is thus T_0.

What is time between ticks in frame K?

Since clock is at origin of K', then position of 1st and 2nd ticks is at $x_1' = x_2' = 0$

In frame K, the observer sees tick 1 at

$$ct_1 = \gamma c t_1' + \gamma\left(\frac{v}{c}\right)x_1' = 0 + 0 = 0$$

and tick 2 at

$$ct_2 = \gamma c t_2' + \gamma\left(\frac{v}{c}\right)x_2' = \gamma c T_0 + 0 = \gamma c T_0$$

So $t_1 = 0$, $t_2 = \gamma T_0$

So time between ticks as seen by K is $\Delta t = \gamma T_0 > T_0$

So it looks to K as if K's clock has slowed down

<u>Proper Time</u> - time between two events as measured in the frame of reference in which those two events occur at the same position.

T_0 is the <u>proper time</u> between ticks of the clock

FitzGerald Contraction

Consider frame K' moving with $v\hat{x}$ as seen by K.
A ruler, at rest in K', has it's ends located at
$x_1' = 0$, $x_2' = L_0$. What is the length of the ruler as seen by K? Recall, when origins of K and K' coincide then $t = t' = 0$.

At $t=0$ in frame K, the observer measures the positions of the two ends of the ruler and finds

$$x_1' = 0 = -\gamma\left(\frac{v}{c}\right)ct_1 + \gamma x_1 = \underset{t_1=0}{0} + \gamma x_1$$

$$\Rightarrow x_1 = 0$$

and

$$x_2' = L_0 = -\gamma\left(\frac{v}{c}\right)ct_2 + \gamma x_2 = \underset{t_2=0}{0} + \gamma x_2$$

$$\Rightarrow x_2 = \frac{L_0}{\gamma}$$

$t_1 = t_2$ since observer in K measures ends of rod "at the same time"

So length of ruler in K is $x_2 - x_1 = \frac{L_0}{\gamma} < L_0$

It appears to K as if the ruler has ~~sho~~ contracted.

Proper Length - distance between two events as measured in the frame in which the two events happen at the same time