As seen by an observer in frame K'

<u>Note</u>: K's measurement of left end occurs at time

$$ct_1' = \gamma c t_1 - \gamma\left(\frac{v}{c}\right)x_1 = 0 \quad \Rightarrow t_1' = 0$$

K's measurement of right end occurs at time

$$ct_2' = \gamma c t_2 - \gamma\left(\frac{v}{c}\right)x_2 = 0 - \gamma\left(\frac{v}{c}\right)\frac{L_0}{\gamma} = -\frac{v}{c}L_0$$

$$t_2' = -\frac{v}{c^2}L_0$$

$t_1 = t_2 = 0$ as K measures the ends of the rod "at the same time"

So K's interpretation of K's measurement is that K first measures the position of the right end of the ruler, and only a time $\frac{v}{c^2}L_0$ later measures the location of the left end.

So K' sees K measure a length

$$L_0 - \underbrace{\frac{v^2}{c^2}L_0}_{\text{distance ruler travels between K's two measurements}}$$

$$= L_0\left(1 - \frac{v^2}{c^2}\right) = \frac{L_0}{\gamma^2}$$

So K's two measurements, which are simultaneous to K, do not occur simultaneously to K'.

Events that are simultaneous in one frame of reference are not simultaneous in another frame of reference

So K' sees K measure a length that is according to K' a length equal to $\frac{L_0}{\gamma^2}$

But K' also sees that K is measuring with a ~~a ruler~~ length scale that is ~~...~~ Fitzgerald contracted by a factor $1/\gamma$. So the length $\frac{L_0}{\gamma^2}$ seen by K' looks like the length

$\left(\frac{L_0}{\gamma^2}\right)\frac{1}{(1/\gamma)}$ when K' sees K measure it with

K's contracted rulers. Thus K' will agree that K thinks the ruler is $\frac{L_0}{\gamma^2}\gamma = \frac{L_0}{\gamma}$ long.

K thinks the moving ruler has contracted

K' thinks K is both (i) not measuring the ends of the ruler at the same time, and (ii) measuring the length of K''s ruler with K's contracted ruler.

So they can both agree on the outcome of what happens, but they ascribe different physical processes to what is happening.

proper time

two events $\left\{ \begin{array}{l} (x_1, t_1) \\ (x_2, t_2) \end{array} \right\}$ seen in K

Transform to frame K' in which they are at same position $x_1' = x_2'$. The time $t_2' - t_1'$ in that frame K' is the proper time between the events

$$ct_1' = \gamma c t_1 - \gamma\left(\tfrac{v}{c}\right) x_1$$
$$ct_2' = \gamma c t_2 - \gamma\left(\tfrac{v}{c}\right) x_2$$

$$x_1' = -\gamma\left(\tfrac{v}{c}\right) c t_1 + \gamma x_1$$
$$x_2' = -\gamma\left(\tfrac{v}{c}\right) c t_2 + \gamma x_2$$

$$x_1' = x_2' \Rightarrow \gamma(x_2 - x_1) - \gamma\left(\tfrac{v}{c}\right) c\,(t_2 - t_1) = 0$$

$$\Rightarrow \frac{x_2 - x_1}{t_2 - t_1} = v$$

So frame K' travels with $v\hat{x}$ with respect to K.
clearly can have such a K' only if $v < c$.
ie $x_2 - x_1 < c(t_2 - t_1)$ for the two points to be timelike

proper time

The time difference between the events in K' is

$$t_2' - t_1' = \gamma t_2 - \gamma \frac{v}{c^2} x_2 - \gamma t_1 + \gamma \frac{v}{c^2} x_1$$

$$= \gamma\left(t_2 - t_1 - \frac{v}{c^2}(x_2 - x_1)\right)$$

$$= \gamma\left(t_2 - t_1 - \frac{v^2}{c^2}\cdot(t_2 - t_1)\right)$$

$$= (t_2 - t_1)\gamma\left(1 - v^2/c^2\right) = (t_2 - t_1)\,\gamma/\gamma^2$$

$$\boxed{\tau \equiv t_2' - t_1' = \frac{t_2 - t_1}{\gamma}}$$

t_2, t_1 times in frame K
v transforms to frame in which $x_1' = x_2'$

Proper length

two events (x_1, t_1) (x_2, t_2) seen in K
transform to K' in which they occur at same time
$t_1' = t_2'$. The distance $x_2' - x_1'$ in that frame K'
is the <u>proper length</u> between the two events

$$x_1' = -\gamma\left(\frac{v}{c}\right)ct_1 + \gamma x_1$$
$$x_2' = -\gamma\left(\frac{v}{c}\right)ct_2 + \gamma x_2$$

$$ct_1' = \gamma ct_1 - \gamma\left(\frac{v}{c}\right)x_1$$
$$ct_2' = \gamma ct_2 - \gamma\left(\frac{v}{c}\right)x_2$$

$$t_1' = t_2' \Rightarrow \gamma c(t_2 - t_1) - \gamma\left(\frac{v}{c}\right)(x_2 - x_1) = 0$$

$$\frac{x_2 - x_1}{t_2 - t_1} = \frac{c^2}{v}$$

or $v = \dfrac{c^2(t_2 - t_1)}{(x_2 - x_1)}$ such a frame K' can exist only if $v < c$ or $\dfrac{x_2 - x_1}{t_2 - t_1} = \dfrac{c^2}{v} > c$

$x_2 - x_1 > c(t_2 - t_1)$ for the two points to be <u>space-like</u>

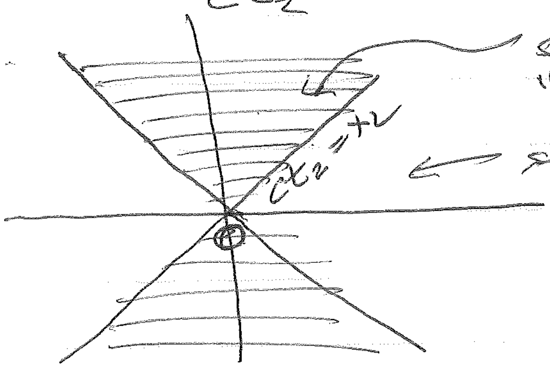
Then the proper length is

$$\ell \equiv x_2' - x_1' = \gamma(x_2 - x_1) - \gamma\left(\frac{v}{c}\right)c(t_2 - t_1)$$
$$= \gamma(x_2 - x_1) - \gamma\left(\frac{v}{c}\right)c\,\frac{v}{c^2}(x_2 - x_1)$$
$$= (x_2 - x_1)\gamma\left(1 - \frac{v^2}{c^2}\right) = (x_2 - x_1)\gamma/\gamma^2$$

$$\boxed{\ell = \frac{x_2 - x_1}{\gamma}}$$

x_2, x_1 positions in frame K
v transforms to frame in which $t_1' = t_2'$

Consider two events, one of which occurs at $(x_1 = 0, t_1 = 0)$ and the other at (x_2, t_2)



set of points such that $\frac{x_2}{t_2} < c$ "time-like" region

set of points such that $\frac{x_2}{t_2} > c$ "space-like" region

The time-like region $\frac{x_2}{t_2} < c$ consists of all points such that there is a frame in which x_2 occurs at the same position as x_1 and we can therefore define the proper time between the two events.
Time-like region is such that a pulse of light emitted at origin at $t_1 = 0$ will arrive at position x_2 at a time earlier than t_2.

The space-like region $\frac{x_2}{t_2} > c$ consists of all points such that there is a frame in which t_2 occurs at the same time as t_1 and we can therefore define the proper length between the two events
Space-like region is such that a pulse of light emitted at origin at $t_1 = 0$ will arrive at position x_2 at a time later than t_2

The light cone $\frac{x_2}{t_2} = c$ separates the time-like from the space-like regions. The pt at origin can effect only events in its future time-like region It is effected only by events in its past time-like region

~~Inverse transform obtained by taking $v \to -v$ in above~~

$$\cancel{\begin{cases} ct = \gamma ct' + \gamma\left(\frac{v}{c}\right) x' \\ x = \gamma\left(\frac{v}{c}\right) ct' + \gamma x' \end{cases}}$$

4 - vectors

4-position: $x_\mu = (x_1, x_2, x_3, ict)$ $\qquad x_4 \equiv ict$

summation convention $x_\mu x_\mu \equiv \sum_{\mu=1}^{4} x_\mu^2 = r^2 - c^2t^2$ Lorentz invariant scalar
- sum over repeated indices — has same value in all interial frames

Lorentz transf for $K \to K'$ where K' moves with $v\hat{x}$ as seen by K.

$$\left.\begin{aligned} x_1' &= \gamma\left(x_1 + i\left(\tfrac{v}{c}\right)x_4\right) \\ x_2' &= x_2 \\ x_3' &= x_3 \\ x_4' &= \gamma\left(x_4 - i\left(\tfrac{v}{c}\right)x_1\right) \end{aligned}\right\}$$

linear transf, can be represented by a matrix

or $x_\mu' = a_{\mu\nu}(L)\, x_\nu$

↑ matrix of Lorentz transformation L

$$a(L) = \begin{pmatrix} \gamma & 0 & 0 & i\frac{v}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\frac{v}{c}\gamma & 0 & 0 & \gamma \end{pmatrix}$$

inverse: $x_\mu = a_{\mu\nu}(L^{-1})\, x_\nu'$

$a_{\mu\nu}(L^{-1})$ is given by taking $v \to -v$ in $a_{\mu\nu}(L)$

we see $a_{\mu\nu}(L^{-1}) = a_{\nu\mu}(L)$

inverse = transpose ⇒ "orthogonal"

More generally

Since $x_\mu'^2$ is Lorentz invariant scalar,

$$x_\mu'^2 = a_{\mu\nu}(L)\, a_{\mu\lambda}(L)\, x_\nu x_\lambda = x_\lambda^2$$

$$\Rightarrow \quad a_{\mu\nu}(L)\, a_{\mu\lambda}(L) = \delta_{\nu\lambda}$$

$$\Rightarrow \quad a^t_{\nu\mu}(L)\, a_{\mu\lambda}(L) = \delta_{\nu\lambda}$$

$$\Rightarrow \quad a^t_{\mu\nu} = a^{-1}_{\mu\nu}(L) \qquad \text{transpose} = \text{inverse}$$

$a_{\mu\nu}$ is 4×4 orthogonal matrix

~~a matrix whose transpose equals its inverse~~ is called an orthogonal matrix.

If L_1 is a Lorentz transf from K to K'

L_2 is a Lorentz transf from K' to K''

Then the Lorentz transf from K to K'' is given by the matrix

$$a(L_2 L_1) = a(L_2)\, a(L_1)$$

if $L_1 \equiv L$ and $L_2 = L^{-1}$ so $L_2 L_1 = I$ identity

$$\Rightarrow \quad a^{-1}(L) = a(L^{-1})$$

particle on trajectory $\vec{r}(t)$

$dx_1 = x_1(t+dt) - x_1(t)$

etc

4-differential

$$dx_\mu = (dx_1, dx_2, dx_3, icdt)$$

$$-(dx_\mu)^2 \equiv c^2 ds^2 = c^2 dt^2 - dr^2 \qquad \text{Lorentz invariant scalar}$$

$$ds^2 = dt^2\left[1 - \frac{1}{c^2}\left(\frac{dx_1}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dx_2}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dx_3}{dt}\right)^2\right]$$

$$ds^2 = \frac{dt^2}{\gamma^2} \qquad \vec{v} = \frac{d\vec{r}}{dt}$$

$$\boxed{ds = \frac{dt}{\gamma}}$$

proper time interval

ds is the same in all inertial frames.

A 4-vector is any 4 numbers that transform under a Lorentz transformation the same way as does x_μ

<u>4-velocity</u> $\quad u_\mu \equiv \dfrac{dx_\mu}{ds} \equiv \mathring{x}_\mu \quad$ dot indicates derivative with respect to s

$$= \gamma \frac{dx_\mu}{dt}$$

since dx_μ is a 4-vector and ds is Lorentz invarient scalar, then $\frac{dx_\mu}{ds}$ is a 4-vector,

space components $\vec{u} = \gamma \vec{v}$

$u_4 = ic\gamma \qquad u_\mu = \gamma(\vec{v}, ic)$

$$u_\mu u_\mu = \gamma^2 v^2 - c^2\gamma^2 = \gamma^2(v^2 - c^2)$$

$$= \frac{v^2 - c^2}{1 - \frac{v^2}{c^2}} = -c^2 \quad \text{Lorentz invarient scalar}$$

<u>4-acceleration</u> $\quad a_\mu \equiv \dfrac{du_\mu}{ds} = \gamma \dfrac{du_\mu}{dt}$

<u>4-gradient</u> $\quad \dfrac{\partial}{\partial x_\mu} \equiv \left(\vec{\nabla}, \dfrac{-i}{c}\dfrac{\partial}{\partial t}\right) = \left(\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}, \dfrac{\partial}{\partial x_3}, \dfrac{\partial}{\partial x_4}\right)$

where $x_4 = ict$

proof $\dfrac{\partial}{\partial x_\mu}$ is a 4-vector

by chain rule: $\dfrac{\partial}{\partial x_\mu'} = \dfrac{\partial x_\lambda}{\partial x_\mu'}\dfrac{\partial}{\partial x_\lambda} \longrightarrow$ but $\dfrac{\partial x_\lambda}{\partial x_\mu'} = a_{\lambda\mu}(L^{-1})$

$= a_{\mu\lambda}(L)$

inverse = transpose

So $\dfrac{\partial}{\partial x_\mu'} = a_{\mu\lambda}(L)\dfrac{\partial}{\partial x_\lambda}$

so transforms same as x_μ

$$\left(\frac{\partial}{\partial x_\mu}\right)^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \quad \text{wave equation operator!}$$

<u>inner products</u>

If U_μ and V_μ are 4-vectors, then $U_\mu V_\mu$ is Lorentz invarient scalar