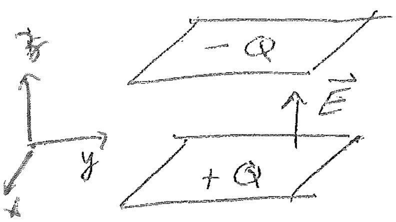
Maxwell's Equations in Relativistic Form

How do $\vec{E}$ and $\vec{B}$ transform under Lorentz transformation?
$\vec{E}$ and $\vec{B}$ have much more complicated transformation laws than position 4-vector $x_\mu = (\vec{r}, ict)$

Example: parallel plate capacitor at rest in K
plates have area A, charge Q:



$\vec{E} = \dfrac{Q}{A\varepsilon_0}\hat{z}$ uniform $\qquad \dfrac{Q}{A} = \sigma$ surface charge den

$\vec{B} = 0$

In K', moving with $\vec{v} = v\hat{y}$ wrt K, y dimension of plates is contracted by factor $1/\gamma$ (FitzGerald Contraction)

$$\sigma' = \frac{Q}{A'} = \frac{\gamma Q}{A} = \gamma\sigma$$

Assume Q is a Lorentz invarient scalar

$$\vec{E}' = \frac{Q}{A'\varepsilon_0}\hat{z} = \frac{\gamma Q}{A\varepsilon_0}\hat{z} = \gamma\vec{E}$$

$\vec{E}$ is along $\hat{z} \perp \vec{v}$.

This is different than transf law for $\vec{r}$.
Under L.T. components of $\vec{r} \perp \vec{v}$ do not change
But components of $\vec{E} \perp \vec{v}$ do change

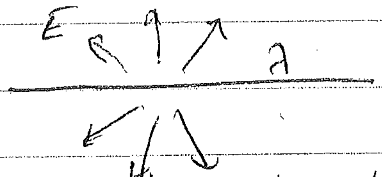
Also, moving surface charge σ' gives rise to surface current density $\Rightarrow$ there will be magnetic field $\vec{B}'$ in frame K'. $\rightarrow$ Lorentz transf must couple together the components of $\vec{E}$ and $\vec{B}$.

Electromagnetism

Clearly $\vec{E}$ + $\vec{B}$ must transform into each other under Lorentz transf.

in inertial frame K

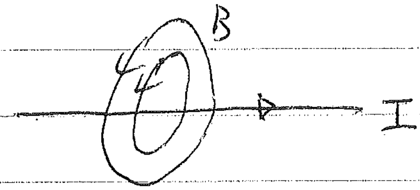
stationary line charge λ



cylindrical outward electric field

no B-field

in frame K' moving with $\vec{v}$ ∥ to wire



moving line charge gives current $\Rightarrow$ B circulating around wire as well as outward radial E

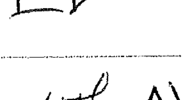
Lorentz force

$$\vec{F} = q\vec{E} + q\,\vec{v}\times\vec{B}$$

What is the velocity $\vec{v}$ here? velocity with respect to what inertial frame? clearly $\vec{E}$ and $\vec{B}$ must change from one inertial frame to another if this force law can make sense.

Charge density, current density

ΔV contains charge ΔQ



Consider charge ΔQ contained in a vol ΔV.

ΔQ is a Lorentz invariant scalar.

Consider the reference frame in which the charge is instantaneously at rest. In this frame

$\Delta Q = \overset{\circ}{\rho} \, \Delta \overset{\circ}{V}$ $\quad$ $\overset{\circ}{\rho}$ is charge density in rest frame of charge
$\Delta \overset{\circ}{V}$ is volume of box in rest frame

$\overset{\circ}{\rho}$ is a Lorentz invarient scalar by definition
Now transform to anothe frame moving with velocity $\vec{v}$ with respect to the rest frame.

ΔQ remains the same.
$\Delta V = \dfrac{\Delta \overset{\circ}{V}}{\gamma}$ volume contracts in direction $\parallel$ to $\vec{v}$

$$\Rightarrow \rho = \frac{\Delta Q}{\Delta V} = \frac{\Delta Q}{\Delta \overset{\circ}{V}} \gamma = \overset{\circ}{\rho} \gamma$$

spatial components of 4-velocity

current density is $\vec{j} = \rho \vec{v} = (\rho/\gamma)(\gamma \vec{v}) = \overset{\circ}{\rho} \, \vec{u}$

Define 4-current $j_\mu \equiv \overset{\circ}{\rho} \, u_\mu = \overset{\circ}{\rho} (\vec{u}, ic\gamma)$

spatial components of j_μ are $\vec{j} = \overset{\circ}{\rho} \vec{u} = \rho \vec{v}$ current density

temporal component of j_μ is $j_4 = ic \overset{\circ}{\rho} \gamma = ic\rho$ charge density

So $\boxed{j_\mu = (\vec{j}, ic\rho)}$ $\quad$ j_μ is a 4-vector since u_μ is a 4-vector and $\overset{\circ}{\rho}$ is Lorentz invarient scalar

length of the 4-current is $j_\mu j_\mu = |\vec{j}|^2 - c^2 \rho^2 = \overset{\circ}{\rho}^2 u_\mu u_\mu$
$= -c^2 \overset{\circ}{\rho}^2$

Charge conservation

$$0 = \vec{\nabla} \cdot \vec{j} + \frac{\partial \rho}{\partial t} = \vec{\nabla} \cdot \vec{j} + \frac{\partial (ic\rho)}{\partial (ict)} = \vec{\nabla} \cdot \vec{j} + \frac{\partial j_4}{\partial x_4}$$

$\Rightarrow \boxed{\dfrac{\partial j_\mu}{\partial x_\mu} = 0}$ charge conservation in Lorentz covariant form

Equations for potentials in Lorentz gauge

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = \Box^2 \vec{A} = -\mu_0 \vec{J}$$

$$c^2 = \frac{1}{\mu_0 \varepsilon_0}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V = \Box^2 V = -\rho/\varepsilon_0 = -c^2\mu_0\rho$$

$$= -\mu_0 (ic\rho)\left(\frac{c}{i}\right)$$

$$= -\mu_0 j_4 \left(\frac{c}{i}\right)$$

So

$$\Box^2 \vec{A} = -\mu_0 \vec{J}$$

$$\Box^2 (iV/c) = -\mu_0 j_4$$

Define 4-potential $\boxed{A_\mu = (\vec{A}, iV/c)}$

$$\Rightarrow \quad \Box^2 A_\mu = -\mu_0 j_\mu \quad \text{equation for potentials}$$

$\Box^2 = \dfrac{\partial^2}{\partial x_\nu^2}$ is Lorentz invariant operator

So we can write the above as $\boxed{\dfrac{\partial^2 A_\mu}{\partial x_\nu^2} = -\mu_0 j_\mu}$

Lorentz gauge condition is

$$0 = \vec{\nabla}\cdot\vec{A} + \mu_0\varepsilon_0 \frac{\partial V}{\partial t} = \vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}$$

$$= \vec{\nabla}\cdot\vec{A} + \frac{\partial (iV/c)}{\partial (ict)} = \vec{\nabla}\cdot\vec{A} + \frac{\partial A_4}{\partial x_4}$$

$$= \frac{\partial A_\mu}{\partial x_\mu}$$

So Lorentz Gauge condition is $\boxed{\dfrac{\partial A_\mu}{\partial x_\mu} = 0}$

Electric and Magnetic Fields

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \Rightarrow \quad \boxed{B_i = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}}$$

where i, j, k are a cyclic permutation of 1,2,3

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

$$V = \frac{cA_4}{i}, \quad x_4 = ict$$

$$\Rightarrow E_i = -\frac{\partial\left(\frac{c}{i}A_4\right)}{\partial x_i} - \frac{\partial A_i}{\partial\left(\frac{x_4}{ic}\right)} = -\frac{c}{i}\frac{\partial A_4}{\partial x_i} - ic\frac{\partial A_i}{\partial x_4}$$

$$\boxed{\frac{E_i}{c} = i\left(\frac{\partial A_4}{\partial x_i} - \frac{\partial A_i}{\partial x_4}\right)}$$

has a similar form to B_i

We define the field strength tensor

$$\boxed{F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}} = -F_{\nu\mu}$$

4x4 antisymmetric 2nd rank tensor

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1/c \\ -B_3 & 0 & B_1 & -iE_2/c \\ B_2 & -B_1 & 0 & -iE_3/c \\ iE_1/c & iE_2/c & iE_3/c & 0 \end{pmatrix}$$

"curl" of a 4-vector is a 4x4 antisymmetric 2nd rank tensor

4x4 antisymmetric 2nd rank tensor has only 6 independent components – just the right number to specify the $\vec{E}$ and $\vec{B}$ fields!

$F_{\mu\nu}$ transforms under a Lorentz transformation just like a tensor (i.e. not like a vector)

$$F'_{\mu\nu} = \frac{\partial A'_\nu}{\partial x'_\mu} - \frac{\partial A'_\mu}{\partial x'_\nu} \qquad \text{use } \left.\begin{array}{l} A'_\nu = a_{\nu\lambda} A_\lambda \\ \dfrac{\partial}{\partial x'_\mu} = a_{\mu\sigma} \dfrac{\partial}{\partial x_\sigma} \end{array}\right\} \text{ since } A_\mu \text{ and } \frac{\partial}{\partial x_\sigma} \text{ are both 4-vectors}$$

$$F'_{\mu\nu} = a_{\nu\lambda} a_{\mu\sigma} \frac{\partial A_\lambda}{\partial x_\sigma} - a_{\mu\sigma} a_{\nu\lambda} \frac{\partial A_\sigma}{\partial x_\lambda}$$

$$= a_{\mu\sigma} a_{\nu\lambda} \left(\frac{\partial A_\lambda}{\partial x_\sigma} - \frac{\partial A_\sigma}{\partial x_\lambda} \right)$$

$$\boxed{F'_{\mu\nu} = a_{\mu\sigma} a_{\nu\lambda} F_{\sigma\lambda}} \leftarrow \text{transformation law for a 2nd rank tensor}$$

In terms of matrix multiplication, and writing for the transpose of a matrix $a_{\nu\lambda} = a^t_{\lambda\nu}$, the above can b written as

$$F'_{\mu\nu} = a_{\mu\sigma} F_{\sigma\lambda} a^t_{\lambda\nu}$$

The above has the form of the product of three matrices

If we write out the above transformation law component by component we get the following transformation law for the $\vec{E}$ and $\vec{B}$ fields.

For a transformation from K to K', where K' moves with velocity $v\hat{x}$ as seen from K,

$$E'_1 = E_1 \qquad\qquad B'_1 = B_1$$

$$E'_2 = \gamma(E_2 - vB_3) \qquad\qquad B'_2 = \gamma\left(B_2 + \frac{v}{c^2} E_3\right)$$

$$E'_3 = \gamma(E_3 + vB_2) \qquad\qquad B'_3 = \gamma\left(B_3 - \frac{v}{c^2} E_2\right)$$

where $(1,2,3) = (x,y,z)$

The transformation law for an n^{th} rank tensor is

$$T'_{\mu_1, \mu_2, \ldots \mu_n} = a_{\mu_1 \nu_1} a_{\mu_2 a_2} \cdots a_{\mu_n \nu_n} T_{\nu_1, \nu_2, \ldots, \nu_n}$$

Inhomogeneous Maxwell's Equations

Using the field strength tensor $F_{\mu\nu}$ we can write the inhomogeneous Maxwell's equations (ie the ones involving the sources ρ and $\vec{J}$) as follows:

$$\boxed{\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 J_\mu}$$

$F_{\mu\nu}$ is a 4-tensor 2nd rank

$\frac{\partial}{\partial x_\nu}$ is a 4-vector

$\Rightarrow \frac{\partial F_{\mu\nu}}{\partial x_\nu}$ is a 4-vector

Proof that $\frac{\partial F_{\mu\nu}}{\partial x_\nu}$ is a 4-vector. Using the transformation laws of $F_{\mu\nu}$ and $\frac{\partial}{\partial x_\nu}$ we get

$$\frac{\partial F'_{\mu\nu}}{\partial x'_\nu} = a_{\mu\lambda} a_{\nu\sigma} a_{\nu\tau} \frac{\partial F_{\lambda\sigma}}{\partial x_\tau}$$

write $\sum_\nu a_{\nu\sigma} a_{\nu\tau} = \sum_\nu a^t_{\sigma\nu} a_{\nu\tau}$

but since a is orthogonal, $a^t = a^{-1}$ and $\sum_\nu a^t_{\sigma\nu} a_{\nu\tau} = \delta_{\sigma\tau}$

$$\frac{\partial F'_{\mu\nu}}{\partial x'_\nu} = a_{\mu\lambda} \delta_{\sigma\tau} \frac{\partial F_{\lambda\sigma}}{\partial x_\tau} = a_{\mu\lambda} \frac{F_{\lambda\sigma}}{\partial x_\sigma}$$

so transforms like a 4-vector