back to $\boxed{\dfrac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 j_\mu}$

To see this is so, substitute in definition of $F_{\mu\nu}$ in terms of 4-potential $A_\mu$

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{\partial}{\partial x_\nu}\left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\right) = \frac{\partial}{\partial x_\mu}\left(\frac{\partial A_\nu}{\partial x_\nu}\right) - \frac{\partial^2 A_\mu}{\partial x_\nu^2}$$

1st term = 0 by Lorentz gauge condition. So

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = -\frac{\partial^2 A_\mu}{\partial x_\nu^2} = -\Box^2 A_\mu = \mu_0 j_\mu$$

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 j_\mu \Rightarrow \begin{cases} \vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j} & \text{spatial components} \\ \vec{\nabla}\cdot\vec{E} = \mu_0 c^2 \rho = \rho/\varepsilon_0 & \text{temporal component} \end{cases}$$

We still need to have a Lorentz covariant way to write the homogeneous Maxwell Equations.

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \text{and} \quad \vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

## Homogeneous Maxwell Equations

Construct the 3rd rank co-variant tensor

$$\boxed{\tilde{G}_{\mu\nu\lambda} \equiv \frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu}}$$

transforms as $\tilde{G}'_{\mu\nu\lambda} = a_{\mu\alpha} a_{\nu\beta} a_{\lambda\gamma} \tilde{G}_{\alpha\beta\gamma}$

$\tilde{G}_{\mu\nu\lambda}$ has in principle $4^3 = 64$ components

But can show that $\tilde{G}$ is antisymmetric in exchange of any two indicies

$$\tilde{G}_{\nu\mu\lambda} = \frac{\partial F_{\nu\mu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\nu}}{\partial x_\mu} + \frac{\partial F_{\mu\lambda}}{\partial x_\nu} \qquad \text{but since } F_{\mu\nu} = -F_{\nu\mu}$$

$$= -\frac{\partial F_{\mu\nu}}{\partial x_\lambda} - \frac{\partial F_{\nu\lambda}}{\partial x_\mu} - \frac{\partial F_{\lambda\mu}}{\partial x_\nu} = -G_{\mu\nu\lambda}$$

$\Rightarrow$ $\tilde{G}_{\nu\mu\lambda} = 0$ if any two indicies are equal

$\Rightarrow$ there are only $\underline{\underline{4}}$ independent components of $G_{\mu\nu\lambda}$
these are

$$\tilde{G}_{123}, \tilde{G}_{124}, \tilde{G}_{134}, \tilde{G}_{234}$$

all other components are just equal to $\pm$ one of these according to permutation of indicies.

The 4 homogeneous Maxwell equations can be written as

$$\boxed{\tilde{G}_{\mu\nu\lambda} = 0}$$

To see that above is true, substitute in for $F_{\mu\nu}$ in terms of potential $A_\mu$ in definition of $\tilde{G}$

$$\tilde{G}_{\mu\nu\lambda} = \underbrace{\frac{\partial^2 A_\nu}{\partial x_\lambda \partial x_\mu} \underbrace{- \frac{\partial^2 A_\mu}{\partial x_\lambda \partial x_\nu} + \frac{\partial^2 A_\mu}{\partial x_\nu \partial x_\lambda}}_{\text{cancell}} \underbrace{- \frac{\partial^2 A_\lambda}{\partial x_\nu \partial x_\mu} + \frac{\partial^2 A_\lambda}{\partial x_\mu \partial x_\nu}}_{\text{cancell}} - \frac{\partial^2 A_\nu}{\partial x_\mu \partial x_\lambda}}_{\text{cancell}}$$

also, one has

$$\tilde{G}_{123} = \frac{\partial F_{12}}{\partial x_3} + \frac{\partial F_{31}}{\partial x_2} + \frac{\partial F_{23}}{\partial x_1} = \frac{\partial B_3}{\partial x_3} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_1}{\partial x_1} = 0$$

$$\tilde{G}_{123} = 0 \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{B} = 0$$

$$\tilde{G}_{412} = \frac{\partial F_{41}}{\partial x_2} + \frac{\partial F_{24}}{\partial x_1} + \frac{\partial F_{12}}{\partial x_4} = \frac{i\,\partial E_1}{c\,\partial x_2} + \frac{-i\,\partial E_2}{c\,\partial x_1} + \frac{\partial B_3}{ic\,\partial t}$$

$$= \frac{i}{c}\left[\frac{\partial E_1}{\partial x_2} - \frac{\partial E_2}{\partial x_1} - \frac{\partial B_3}{\partial t}\right] = -\frac{i}{c}\left[(\vec{\nabla}\times\vec{E})_3 + \frac{\partial B_3}{\partial t}\right] = 0$$

this is the z-component of Faraday's law $\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$

$\tilde{G}_{413} = 0$ and $\tilde{G}_{423} = 0$ give x and y components of Faraday's law.

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An alternative way to write the homogeneous Maxwell's Equations

Note: we can get the homogeneous Maxwell's equations from the inhomogeneous equations by making the substitutions

$$\vec{j} \to 0, \quad \rho \to 0, \quad \frac{\vec{E}}{c} \to \vec{B}, \quad \vec{B} \to -\frac{\vec{E}}{c}$$

So we define the <u>dual field strength tensor</u>

$$G_{\mu\nu} = \begin{pmatrix} 0 & -E_3/c & E_2/c & -iB_1 \\ E_3/c & 0 & -E_1/c & -iB_2 \\ -E_2/c & E_1/c & 0 & -iB_3 \\ iB_1 & iB_2 & iB_3 & 0 \end{pmatrix}$$

or equivalently if

$$\varepsilon_{\mu\nu\sigma\lambda} = \begin{cases} +1 & \text{if } \mu\nu\sigma\lambda \text{ is an even permutation of } 1234 \\ -1 & \text{if } \mu\nu\sigma\lambda \text{ is an odd permutation of } 1234 \\ 0 & \text{otherwise, ie any two indeces equal} \end{cases}$$

generalization of the Levi-Civita symbol

then $$\boxed{G_{\mu\nu} = \frac{1}{2i} \varepsilon_{\mu\nu\lambda\sigma} F_{\lambda\sigma}}$$

psuedo-tensor gives wrong sign under parity transform.

Then $$\boxed{\frac{\partial G_{\mu\nu}}{\partial x_\nu} = 0}$$

gives the homogeneous Maxwell's equations

From $F_{\mu\nu}$ ad $G_{\mu\nu}$ we can construct the following Lorentz invariant scalars

$$\frac{1}{2} F_{\mu\nu} F_{\mu\nu} = B^2 - \frac{E^2}{c^2}$$

$$-\frac{1}{4} F_{\mu\nu} G_{\mu\nu} = \frac{\vec{B} \cdot \vec{E}}{c}$$

these have the same value in any inertial frame of reference!

$\Rightarrow$ 1) If $\vec{E} \perp \vec{B}$ ad $|\vec{B}| = \frac{|\vec{E}|}{c}$ in one frame of reference, then it is so in all frames of reference.

$(\vec{E} \cdot \vec{B} = 0$ ad $|\vec{B}|^2 - \frac{|\vec{E}|^2}{c^2} = 0)$

This property is satisfied by EM waves in the vacuum

2) If in one frame $\vec{E} \cdot \vec{B} = 0$ <u>and</u> $\frac{E^2}{c^2} > B^2$, then there exists a frame in which $\vec{B}' = 0$. If in one frame $\vec{E} \cdot \vec{B} = 0$ and $B^2 > E^2/c^2$, then there exists a frame in which $\vec{E}' = 0$.

# Relativistic Kinematics

4-momentum of a particle $\quad p_\mu = m\dot{x}_\mu = m u_\mu = (m\gamma\vec{v}, imc\gamma)$

[ $m$ is mass of particle as measured in the frame in which the particle is instantaneously at rest. $m$ = "rest mass"

$p_\mu$ is a 4-vector since $m$ is a scalar and $u_\mu$ is a 4-vector

$$p_\mu^2 = m^2 u_\mu^2 = -m^2c^2 \qquad \text{since } u_\mu^2 = -c^2$$

4-force $\quad K_\mu = (\vec{K}, iK_0)$ $\quad$ also called "Minkowski force"

We guess that the relativistic generalization of Newton's 2nd law of motion is

$$m\frac{d^2x_\mu}{ds^2} = K_\mu \quad \text{or} \quad m\frac{du_\mu}{ds} = K_\mu$$

$$\text{or} \quad \frac{dp_\mu}{ds} = K_\mu \qquad (p_\mu = m u_\mu = m\dot{x}_\mu)$$

Now since $p_\mu^2 = -m^2c^2$ is a constant, we have

$$0 = \frac{d}{ds}(p_\mu^2) = 2p_\mu\frac{dp_\mu}{ds} = 2p_\mu K_\mu$$

$$\Rightarrow p_\mu K_\mu = 0$$

$$p_\mu K_\mu = m\gamma\vec{v}\cdot\vec{K} - mc\gamma K_0 = 0$$

$$\text{so} \quad \boxed{K_0 = \frac{\vec{v}}{c}\cdot\vec{K}}$$

Time component of 4-force is determined by the spacial components $\vec{K}$

Define the usual 3-force by

$$\frac{d\vec{p}}{dt} = \vec{F}$$ (we identify the Newtonian momentum $\vec{p}$ with the spatial components of $p_\mu$)

$$\frac{d\vec{p}}{ds} = \vec{K}$$ spatial part of relativistic Newton's law

$$\frac{d\vec{p}}{ds} = \gamma \frac{d\vec{p}}{dt} = \gamma \vec{F}$$ since $ds = dt/\gamma$

$$\Rightarrow \boxed{\vec{K} = \gamma \vec{F}}$$ relation between spatial part of 4-force and the usual 3-force $\vec{F}$

$$\Rightarrow K_0 = \frac{\vec{v}}{c}\cdot\vec{K} = \gamma\frac{\vec{v}}{c}\cdot\vec{F}$$

Consider now the 4-th component of Newton's equation

$$\frac{dp_4}{ds} = m\frac{du_4}{ds} = m\frac{d}{ds}(ic\gamma) = iK_0 = i\gamma\frac{\vec{v}}{c}\cdot\vec{F}$$

$$\Rightarrow \frac{d}{ds}(m\gamma c^2) = \gamma\vec{v}\cdot\vec{F}$$

$$d(m\gamma c^2) = \gamma\vec{v}\cdot\vec{F}\, ds = \gamma\vec{v}\cdot\vec{F}\,\frac{dt}{\gamma}$$

$$= \vec{v}\cdot\vec{F}\, dt = d\vec{r}\cdot\vec{F}$$

$\Rightarrow$

Work-energy theorem: $d(m\gamma c^2) = d\vec{r}\cdot\vec{F}$

work done on particle

$\Rightarrow$ change in kinetic energy of particle

relativistic kinetic energy $\boxed{E = m\gamma c^2}$