$$p_4 = i m\gamma c = iE/c$$

$$p_\mu = (\vec{p}, \tfrac{iE}{c})$$ momentum-energy 4-vector

$$\vec{p} = m\gamma\vec{v}$$

$$E = m\gamma c^2$$

For particles moving at non-relativistic speeds $v \ll c$

$$E = m\gamma c^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}} \simeq \frac{mc^2}{1-\frac{v^2}{2c^2}} \simeq mc^2\left(1+\frac{v^2}{2c^2}\right)$$

$$\simeq mc^2 + \tfrac{1}{2}mv^2$$

non-relativistic kinetic energy

rest mass energy

$\dfrac{dp_\mu}{ds} = K_\mu$ is therefore both the relativistic analogue of Newton's 2nd law, but also the law of conservation of energy (ie the work-energy theorem)

Conservation of momentum and energy

① why is relativistic momentum ~~[illegible]~~ $\vec{p} = m\gamma\vec{v}$ and not just $m\vec{v}$ as in non-relativistic case?

Because we want momentum to be conserved in all frames of reference, $\vec{p}$ must be the spatial part of a 4-vector. We see this as follows.

Suppose momentum was $m\vec{v}$. For a collection of particles, conservation of momentum would mean

$$(*) \qquad \sum_i m_i \vec{v}_i(t_1) = \sum_i m_i \vec{v}_i(t_2)$$

for any times t_1 and t_2

If $(*)$ holds in one frame of reference K, and we now transform to another frame of reference K' moving with velocity $\vec{w}$ wrt K, we would find that in K', $(*)$ is no longer satisfied

$$\text{ie} \quad \sum_i m_i \vec{v}_i{}'(t_1) \neq \sum_i m_i \vec{v}_i{}'(t_2)$$

$\vec{v}_i{}'$ related to $\vec{v}_i$ and $\vec{w}$ via relativistic law for addition of velocities

see Griffiths ~~chapt 10.2~~ example 12.6

However, for the 4-momentum, if

$$P_\mu^{tot}(t_1) = \sum_i P_{\mu i}(t_1) = \sum_i P_{\mu i}(t_2) = P_\mu^{tot}(t_2)$$

in frame K, then $P_\mu'^{tot}(t_1) = P_\mu'^{tot}(t_2)$ in any other frame K', since $P_\mu^{tot}(t_1)$ and $P_\mu^{tot}(t_2)$ both transform the same way under Lorentz transf.

$$p_\mu^{tot}(t_1) = p_\mu^{tot}(t_2)$$

space components $\Rightarrow$ momentum conservation holds in all frames
time component $\Rightarrow$ energy conservation holds in all frames

(2) Why did we write Newton's eqn as $\frac{d\vec{p}}{dt} = \vec{F}$, with $\vec{p} = m\gamma\vec{v}$,

instead of $m\frac{d\vec{v}}{dt} = \vec{F}$ (as if used non-relativistic momentum)

If use $m\frac{d\vec{v}}{dt} = \vec{F}$, then $m\,\vec{v}\cdot\frac{d\vec{v}}{dt} = \vec{v}\cdot\vec{F}$

$$\tfrac{1}{2}m\,d(v^2) = dt\,\vec{v}\cdot\vec{F} = d\vec{r}\cdot\vec{F} = dW$$

$$\Rightarrow \tfrac{1}{2}m\int d(v^2) = \int dW$$

$\frac{1}{2}mv^2 = W$ get non-relativistic kinetic energy

in this formulation, energy W is not the time component of any 4-vector. Therefore if energy was conserved in one frame K, it need not be conserved in another frame K!

Only when we take $\frac{d\vec{p}}{dt} = \vec{F}$ with $\vec{p} = m\gamma\vec{v}$

do we get $\int \vec{F}\cdot d\vec{r} = m\gamma c^2 = p_0 c$ · time component of a 4-vector

$\Rightarrow$ energy conservation holds in all reference frames

Lorentz force in relativistic form

$$\frac{dp_\mu}{ds} = K_\mu$$

What is the K_μ that represents the Lorentz force? And how can we write it in a Lorentz coveriant way?

K_μ should depend on the fields $F_{\mu\nu}$ and on the particle's trajectory x_μ

as $\vec{v} \to 0$ $\vec{K} = q\vec{E}$ (since magnetic force $\to 0$ as $\vec{v} \to 0$)

K_μ can't depend directly on x_μ as the force should be independent of where one puts the origin of the coordinates. So K_μ can depend only on derivatives $\dot{x}_\mu$, $\ddot{x}_\mu$, etc.

As $\vec{v} \to 0$, $\vec{K}$ does not depend on the acceleration, so $\vec{K}$ does not depend on $\ddot{x}_\mu$ or higher derivatives.

So K_μ depends only on $F_{\mu\nu}$ and $\dot{x}_\mu$

We need to form a 4-vector out of $F_{\mu\nu}$ and $\dot{x}_\mu$ that is <u>linear</u> in the fields $F_{\mu\nu}$ and proportional to the charge q. (since we want superposition to hold)

The only possibility is

$$K_\mu = q\, f(\dot{x}_\mu^2)\, F_{\mu\nu}\, \dot{x}_\nu$$

where $f(\mathring{x}_\mu^2)$ is some function of $\mathring{x}_\mu^2$.
But $\mathring{x}_\mu^2 = -c^2$ is a <u>constant</u>, so $f(\mathring{x}_\mu^2)$ is a constant.
That constant, $f(\mathring{x}_\mu^2) = 1$, is determined by the requirement that $\vec{K} = q\vec{E}$ as $\vec{v} \to 0$.

So we have $\boxed{K_\mu = q F_{\mu\nu} \mathring{x}_\nu}$

Let's check what this gives for the ordinary 3-force $\vec{F} = \frac{1}{\gamma}\vec{K}$

ith component $F_i = \frac{1}{\gamma} K_i = \frac{q}{\gamma}\left(\sum_{j=1}^{3} F_{ij}\mathring{x}_j + F_{i4}\mathring{x}_4\right)$

sub in for F_{ij} in terms of $\vec{A}$ $= \frac{q}{\gamma}\left(\sum_{j=1}^{3}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)\mathring{x}_j - \frac{iE_i}{c}(ic\gamma)\right)$

use $\mathring{x}_4 = ic\gamma$

since $F_{i4} = \frac{-iE_i}{c}$

Now $\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} = \varepsilon_{ijk} B_k$

<u>proof:</u> $\varepsilon_{ijk} B_k = \varepsilon_{ijk}\varepsilon_{k\ell m}\frac{\partial A_m}{\partial x_\ell}$ (using ε_{ijk} notation to take $\vec{\partial}\times\vec{A}$)

$= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\frac{\partial A_m}{\partial x_\ell}$

$= \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}$

use $\mathring{x}_j = \gamma v_j$

So $F_i = \frac{q}{\gamma}\sum_{j=1}^{3}\varepsilon_{ijk}B_k\gamma v_j + \frac{q}{\gamma}E_i\gamma$

$= q\sum_{j=1}^{3}\varepsilon_{ijk}B_k v_j + qE_i$

$= qE_i + q(\vec{v}\times\vec{B})_i$

so $\boxed{\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}} = \frac{1}{\gamma}\vec{K}$

The Lorentz force has the same form in all inertial frames.
No relativistic modification is needed

Relativistic Larmor's formula

non relativistic result was

$$P = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2}{c^3} a^2$$

total power radiated by particle with acceleration $\vec{a}$ assuming $v \ll c$

Now consider a particle moving with any speed v.
Consider the inertial frame of reference in which that particle is instantaneously at rest. Call this frame $\mathring{K}$. The velocity in this frame is thus $\mathring{\vec{v}} = 0$, and the charge is at the origin $\mathring{\vec{r}} = 0$.

The power radiated, as seen in the frame $\mathring{K}$, is then exactly

$$\mathring{P} = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2}{c^3} \mathring{a}^2$$

where $\mathring{a}$ is the acceleration in frame $\mathring{K}$.

This result is exact because as $v/c \to 0$ all terms higher than the electric dipole term will vanish.

What we need to do is to find the way to Lorentz transform the result $\mathring{P}$ and find its value in any other frame of reference, in which the particle is moving with any velocity $\vec{v}$.

Consider the momentum-energy 4-vector describing the total momentum and total energy of the electromagnetic fields ~~of the charge~~ radiated by the charge.

in frame $\overset{\circ}{K}$ we can write this as

$$\left(\overset{\circ}{\vec{P}}_{EM} , \frac{i \overset{\circ}{\mathcal{E}}}{c} \right)$$

Now $\overset{\circ}{\vec{P}}_{EM} = \int d^3 \overset{\circ}{r} \; \varepsilon_0 \overset{\circ}{\vec{E}} \times \overset{\circ}{\vec{B}}$.

But since the radiated fields are in the radial direction $\hat{\overset{\circ}{r}}$, when we integrate over all space we find $\overset{\circ}{\vec{P}}_{EM} = 0$.

Alternatively you have from homework, for a charge moving with small velocity $\vec{v}$, $\vec{P}_{EM} = \frac{4}{3} \frac{U}{c^2} \vec{v}$

So when $\vec{v} \to 0$, $\vec{P}_{EM} \to 0$.

So in frame $\overset{\circ}{K}$ the momentum energy 4-vector is

$$\left(0 , \frac{i \overset{\circ}{\mathcal{E}}}{c} \right)$$

In a new frame of reference K that moves with velocity $-\vec{v}$ with respect to $\overset{\circ}{K}$ (in frame K, the charge is moving with velocity $\vec{v}$) the energy in frame K is obtained by the transformation law for 4-vectors

$$\frac{i \mathcal{E}}{c} = \gamma \left(\frac{i \overset{\circ}{\mathcal{E}}}{c} + \frac{i v}{c} \overset{\circ}{P}_{EM1} \right)$$

where $\overset{\circ}{P}_{EM1}$ is component of $\overset{\circ}{\vec{P}}_{EM}$ in direction of $\vec{v}$. But $\overset{\circ}{\vec{P}}_{EM} = 0$

So $\frac{i\mathcal{E}}{c} = \gamma \frac{i\mathring{\mathcal{E}}}{c} \Rightarrow \mathcal{E} = \gamma \mathring{\mathcal{E}}$

similarly, if we take the origins of K and $\mathring{K}$ to coincide at the time when we are measuring the radiated power, then time transforms as

$$t = \gamma \mathring{t} + \frac{v}{c^2}\gamma \mathring{x}_1$$ where $\mathring{x}_1$ is position of charge in direction of $\vec{v}$

But charge is at origin in $\mathring{K}$ so $\mathring{x}_1 = 0$

So $t = \gamma \mathring{t}$ ← (since charge is not moving in $\mathring{K}$, $d\mathring{t}$ is really the proper time ds, so this is the familiar $\frac{dt}{\gamma} = ds$)

The Power radiated in frame K is then

$$P = \frac{d\mathcal{E}}{dt} = \frac{\gamma\, d\mathring{\mathcal{E}}}{\gamma\, d\mathring{t}}$$ transforming $\mathcal{E} = \gamma\mathring{\mathcal{E}}$, $t = \gamma\mathring{t}$

$$= \frac{d\mathring{\mathcal{E}}}{d\mathring{t}} = \mathring{P}$$

So the total radiated power is a Lorentz invariant scalar!

$$\boxed{P = \mathring{P} = \frac{1}{4\pi\varepsilon_0}\frac{2}{3} q \frac{\mathring{a}^2}{c^3}}$$

where $\mathring{a}$ is acceleration of charge in its rest frame

We would like to rewrite P in a way that makes no explicit reference to the frame $\mathring{K}$.
ie we want to write $\mathring{a}^2$ in terms of a Lorentz invariant scalar that may be evaluated in any frame K.