PHY 218 **Midterm Exam** **Spring 2019**

1) [30 points total] You should be able to give a short answer to each part of this question without any detailed calculations needed. [10 pts each part]

a) Explain why it was necessary to use electrodynamics in order to calculate the energy stored up in a set of loops carrying steady, time independent, currents. Why could we not have calculated this from our knowledge of electro- and magnetostatics? You do not need any detailed calculation, just explain in words the main point.

b) Explain what is meant by a gauge transformation in electrodynamics (i.e. the full Maxwell's equations, not just statics). Why are gauge transformations useful? Keep your answer brief.

c) Explain the difference between phase velocity and group velocity of an electromagnetic wave in a dielectric material, and explain what causes this difference?

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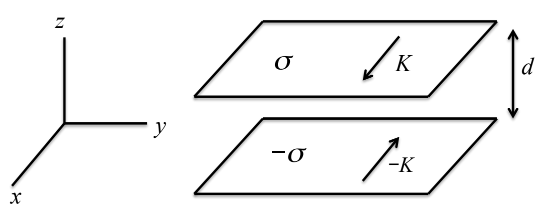
2) [35 points total]

Consider a parallel plane capacitor as in the sketch below. You may regard the planes as infinite. The separation between the planes is d. On the top plane there is a uniform surface charge density σ and a uniform surface current $K\hat{\mathbf{x}}$. On the bottom plane there is a uniform surface charge $-\sigma$ and a uniform surface current $-K\hat{\mathbf{x}}$.



a) [15 pts] Compute the Maxwell stress tensor $\mathbf{T}$ for the regions of space (i) above both planes, (ii) below both planes, (iii) between the two planes. You must show all nine components of the stress tensor.

b) [10 pts] *Using the Maxwell stress tensor*, compute the net force per unit area $\mathbf{f}$ on the top plane. Compute the net force per unit area on the bottom plane. Be sure to give both the magnitude and direction of the forces. Give a simple explanation for the direction of the electric part of this force, and for the magnetic part of this force.

c) [10 pts] For a fixed value of σ, for what value of K will the net force on the top plane vanish? Suppose the surface current on the top plane arises because the plane is moving with velocity $v\hat{\mathbf{x}}$, so $K = \sigma v$. Similarly the surface current on the bottom plane arises because the plane is moving with velocity $-v\hat{\mathbf{x}}$. For what value of v will the force between the planes vanish? Is that physically possible?

(problem 3 on reverse side)

3) [35 points total]

The electric and magnetic fields of a transverse plane electromagnetic wave traveling along the $\hat{\mathbf{z}}$ axis, in a dissipative medium, can be written as:

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left[\mathbf{E}_0 \mathrm{e}^{-k_2 z}\mathrm{e}^{i(k_1 z-\omega t)}\right], \quad \mathbf{B}(\mathbf{r},t) = \mathrm{Re}\left[\frac{|k|}{\omega}(\hat{\mathbf{z}}\times\mathbf{E}_0)\mathrm{e}^{-k_2 z}\mathrm{e}^{i(k_1 z-\omega t+\phi)}\right]$$

where k_1 and k_2 are the real and imaginary parts of the wave vector $k = k_1 + ik_2$, $|k| = \sqrt{k_1^2+k_2^2}$, and $\tan(\phi) = k_2/k_1$. Assume that $\mathbf{E}_0$ is a real valued vector pointing in the xy plane.

For waves in a material, the Poynting vector can be written as

$$\mathbf{S}(\mathbf{r},t) = \frac{1}{\mu}\mathbf{E}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t),$$

where in this problem we will assume that μ is a real valued constant.

a) [10 pts] Compute the instantaneous Poynting vector $\mathbf{S}(\mathbf{r},t)$ for such a wave.

b) [12 pts] Compute the time averaged Poynting vector $\langle\mathbf{S}(\mathbf{r},t)\rangle$. (Your answer should no longer depend on time t!)

c) [6 pts] Compute this time averaged Poynting vector for a frequency ω in a region of strong absorption.

d) [7 pts] Compute this time averaged Poynting vector for a frequency ω in a region of total reflection. Comment on what you find, and say why this is just what you would expect in a region of total reflection.

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Curl: $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem: $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general:

$$\left\{\begin{array}{l} \nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \\ \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \\ \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \end{array}\right.$$

In matter:

$$\left\{\begin{array}{l} \nabla\cdot\mathbf{D} = \rho_f \\ \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \\ \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{H} = \mathbf{J}_f + \frac{\partial\mathbf{D}}{\partial t} \end{array}\right.$$

Auxiliary Fields

Definitions:

$$\left\{\begin{array}{l} \mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{array}\right.$$

Linear media:

$$\left\{\begin{array}{ll} \mathbf{P} = \epsilon_0\chi_e\mathbf{E}, & \mathbf{D} = \epsilon\mathbf{E} \\ \mathbf{M} = \chi_m\mathbf{H}, & \mathbf{H} = \frac{1}{\mu}\mathbf{B} \end{array}\right.$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla\times\mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

Energy, Momentum, and Power

Energy: $U = \frac{1}{2}\int\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau$

Momentum: $\mathbf{P} = \epsilon_0\int(\mathbf{E}\times\mathbf{B})\,d\tau$

Poynting vector: $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$

Larmor formula: $P = \frac{\mu_0}{6\pi c}q^2a^2$

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85\times 10^{-12}\,\mathrm{C^2/Nm^2}$	(permittivity of free space)
$\mu_0 = 4\pi\times 10^{-7}\,\mathrm{N/A^2}$	(permeability of free space)
$c = 3.00\times 10^{8}\,\mathrm{m/s}$	(speed of light)
$e = 1.60\times 10^{-19}\,\mathrm{C}$	(charge of the electron)
$m = 9.11\times 10^{-31}\,\mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\left\{\begin{array}{l} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{array}\right. \qquad \left\{\begin{array}{l} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{array}\right.$$

$$\left\{\begin{array}{l} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right) \\ \phi = \tan^{-1}(y/x) \end{array}\right. \qquad \left\{\begin{array}{l} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{array}\right.$$

Cylindrical

$$\left\{\begin{array}{l} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{array}\right. \qquad \left\{\begin{array}{l} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{array}\right.$$

$$\left\{\begin{array}{l} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{array}\right. \qquad \left\{\begin{array}{l} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{array}\right.$$