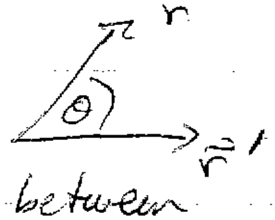
Vectors

position $\vec{r}$ denotes displacement with respect to "origin"

dot product (also called scalar product or inner product)

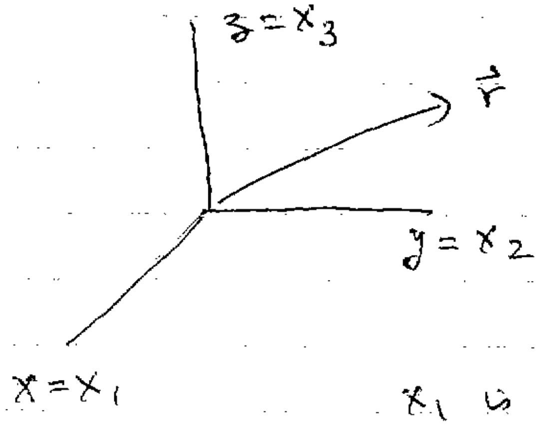
$$\vec{r} \cdot \vec{r}\,' = |\vec{r}|\,|\vec{r}\,'| \cos\theta$$



dot product of two vectors is a scalar — magnitudes — angle between

$$\frac{\vec{r} \cdot \vec{r}\,'}{|\vec{r}\,'|} = |\vec{r}| \cos\theta$$ is projection of $\vec{r}$ onto direction of $\vec{r}\,'$

Rectangular coordinates



$$\vec{r} = x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3$$

$\hat{e}_i$ is unit vector along direction of coordinate

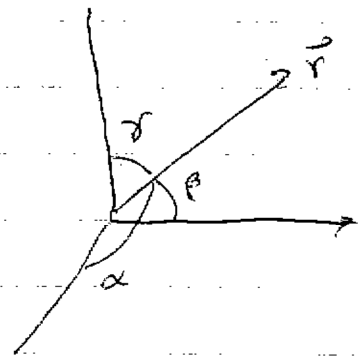
x_1 is projection of $\vec{r}$ onto direction $\hat{e}_1$

$$\Rightarrow \begin{cases} x_1 = \vec{r} \cdot \hat{e}_1 \\ x_2 = \vec{r} \cdot \hat{e}_2 \\ x_3 = \vec{r} \cdot \hat{e}_3 \end{cases}$$

Kronecker delta

follows algebraically from $\hat{e}_i \cdot \hat{e}_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$
since $\hat{e}_i$ are orthogonal

$$\Rightarrow \quad \vec{r} \cdot \hat{e}_1 = (x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3) \cdot \hat{e}_1 = x_1 + 0 + 0$$



α, β, γ are angles $\vec{r}$ makes with respect to $\hat{e}_1, \hat{e}_2, \hat{e}_3$

$\Rightarrow \quad x_1 = r\cos\alpha \qquad\qquad r \equiv |\vec{r}|$

$x_2 = r\cos\beta$

$x_3 = r\cos\gamma$

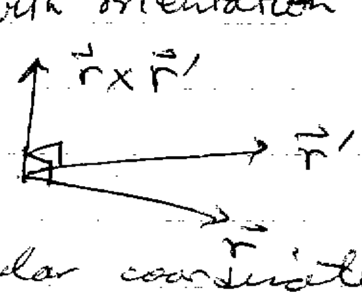
For two vectors $\vec{r}$ and $\vec{r}\,'$

$$\vec{r}\cdot\vec{r}\,' = (x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3)\cdot(x_1'\hat{e}_1 + x_2'\hat{e}_2 + x_3'\hat{e}_3)$$
$$= x_1x_1' + y_1y_1' + z_1z_1' \qquad \text{since } \hat{e}_i\cdot\hat{e}_j = \delta_{ij}$$

<u>cross product</u> - result is a vector

$$|\vec{r}\times\vec{r}\,'| = |\vec{r}||\vec{r}\,'|\sin\theta$$

direction of $\vec{r}\times\vec{r}\,'$ points in direction orthogonal ~~t~~ both $\vec{r}$ and $\vec{r}\,'$, with orientation given by the Right Hand Rule



If write vectors in rectangular coordinates

$$\vec{r}\times\vec{r}\,' = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \end{vmatrix}$$

symbolically expand determinant to get cross product

$$\bar{r} \times \bar{r}' = (x_2 x_3' - x_3 x_2')\,\hat{e}_1 + (x_3 x_1' - x_1 x_3')\,\hat{e}_2 + (x_1 x_2' - x_2 x_1')\,\hat{e}_3$$

$$(\hat{e}_j \times \hat{e}_k)_i \equiv \varepsilon_{ijk} \leftarrow \text{"Levi-Civita density"}$$

$$\varepsilon_{ijk} = \begin{cases} 0 & \text{if any two indices are equal} \\ +1 & \text{if } ijk \text{ even permutation of } 123 \\ -1 & \text{if } ijk \text{ odd permutation of } 123 \end{cases}$$

can check from geometrical definition of cross product that above correctly gives

$$\begin{aligned} \hat{e}_1 \times \hat{e}_2 &= \hat{e}_3 = -\hat{e}_2 \times \hat{e}_1 \\ \hat{e}_2 \times \hat{e}_3 &= \hat{e}_1 = -\hat{e}_3 \times \hat{e}_2 \\ \hat{e}_3 \times \hat{e}_1 &= \hat{e}_2 = -\hat{e}_1 \times \hat{e}_3 \end{aligned}$$

we can now use the above to derive the determinant formulation

$$\begin{aligned} \bar{r} \times \bar{r}' &= (x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3) \times (x_1'\hat{e}_1 + x_2'\hat{e}_2 + x_3'\hat{e}_3) \\ &= x_1 x_2' (\hat{e}_1 \times \hat{e}_2) + x_2 x_1' (\hat{e}_2 \times \hat{e}_1) \\ &\quad + x_1 x_3' (\hat{e}_1 \times \hat{e}_3) + x_3 x_1' (\hat{e}_3 \times \hat{e}_1) \\ &\quad + x_2 x_3' (\hat{e}_2 \times \hat{e}_3) + x_3 x_2' (\hat{e}_3 \times \hat{e}_2) \\ &= (x_1 x_2' - x_2 x_1')\,\hat{e}_3 \\ &\quad + (x_3 x_1' - x_1 x_3')\,\hat{e}_2 \\ &\quad + (x_2 x_3' - x_3 x_2')\,\hat{e}_1 \end{aligned}$$

all terms $\hat{e}_1 \times \hat{e}_1$, $\hat{e}_2 \times \hat{e}_2$, $\hat{e}_3 \times \hat{e}_3 = 0$ in above expansion

In terms of Levi-Civita density

ith component of $\vec{r}\times\vec{r}'$

$$(\vec{r}\times\vec{r}')_i = \sum_{jk} \varepsilon_{ijk}\, x_j\, x_k'$$

$$\vec{r}\cdot\vec{r}' = \sum_i x_i\, x_i'$$

Summation Convention

$$\left.\begin{aligned}(\vec{r}\times\vec{r}')_i &= \varepsilon_{ijk}\, x_j\, x_k' \\ \vec{r}\cdot\vec{r}' &= x_i\, x_i'\end{aligned}\right\}$$ summation over repeated indices is implied

Useful formulae to remember

$\vec{A}\cdot\vec{A} = A^2$ square of length of $\vec{A}$ $\quad A^2 = A_1^2 + A_2^2 + A_3^2$

$\vec{A}\times\vec{A} = 0$

$\vec{A}\cdot(\vec{A}\times\vec{B}) = 0$

$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$

$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}\,(\vec{A}\cdot\vec{C}) - \vec{C}\,(\vec{A}\times\vec{B})$ "BAC − CAB" ru

$$\sum_i \varepsilon_{ijk}\,\varepsilon_{ilm} = \delta_{jl}\,\delta_{km} - \delta_{jm}\,\delta_{kl}$$

can use above to derive BAC − CAB rule

$$\left(\vec{A}\times(\vec{B}\times\vec{C})\right)_i = \sum_{jklm} \varepsilon_{ijk}\, A_j\, \varepsilon_{klm}\, B_l\, C_m$$

$$= \varepsilon_{kij}\,\varepsilon_{klm}\, A_j\, B_l\, C_m \quad \text{as } \varepsilon_{ijk} = \varepsilon_{kij}$$

$$= (\delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl})\, A_j\, B_l\, C_m$$

$$= B_i\,(\vec{A}\cdot\vec{C}) - C_i\,(\vec{A}\cdot\vec{B})$$