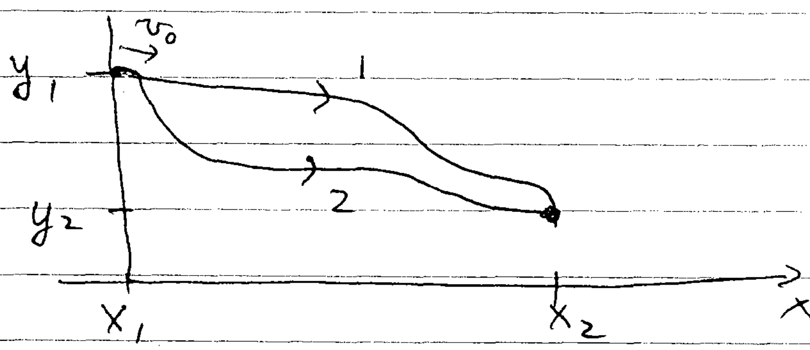
Calculus of Variations

Leads to Lagrangian and Hamiltonian formulations of classical mechanics.

Answers the question of how to find the path that minimizes some specified measure

For example, find the height of a roller coaster track that starts at position (x_1, y_1) and ends at position (x_2, y_2) such that a car which starts at (x_1, y_1) with initial speed v_0 will arrive at (x_2, y_2) in the shortest time



all paths must start at (x_1, y_1) and end at (x_2, y_2). Which gives shortest transit time?

Suppose that the "measure" depends only on the path $y(x)$, its derivative $y'(x) = dy/dx$, and the independent variable x

$$f[\,y(x), y'(x); x\,] \leftarrow$$ input position x, value of function $y(x)$ and its derivative $y'(x)$, $f[y, y'; x]$ outputs a number.

f is a <u>functional</u> of the function $y(x)$.

for example, f might be: $f[y, y'; x] = \frac{1}{2}(y')^2 + gy$

The goal is to find the function $y(x)$ that gives an extremal value (max or min) ~~over~~ of the measure, when integrated between specified end points of the path, i.e. find $y(x)$ that gives extremal value of the integral:

$$J = \int_{x_1}^{x_2} f[y(x), y'(x); x]\, dx$$

where $y(x_1) = y_1$ and $y(x_2) = y_2$ are fixed

Suppose $y(x)$ is such a function. Consider a new function that is "near by" to $y(x)$

$$y_\alpha(x) = y(x) + \alpha\,\eta(x) \qquad \alpha \text{ "small"}$$

where $y(x)$ and $\eta(x)$ are continuous and have continuous first derivatives, and $\eta(x_1) = \eta(x_2) = 0$ (so that $y_\alpha(x_1) = y_1$ and $y_\alpha(x_2) = y_2$)

Clearly, as $\alpha \to 0$, $y_\alpha(x) \to y(x)$.

If $y(x)$ gives the minimum of J, for example, it means that for <u>any</u> function $\eta(x)$, the integral

$$\int f[y + \alpha\eta, y' + \alpha\eta'; x]\, dx$$

gives a <u>larger</u> value, provided α is small enough.

For some particular $\eta(x)$, define

$$J(\alpha) = \int_{x_1}^{x_2} f[y+\alpha\eta,\ y'+\alpha\eta';\ x]\ dx$$

If $y(x)$ gives the extremal value of $J(\alpha)$, then it must be true that $\left.\frac{dJ}{d\alpha}\right|_{\alpha=0} = 0$. Moreover, this must be true for <u>all</u> possible functions $\eta(x)$.

Since the endpoints of integration do not depend on α we have

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta' \right) dx$$

consider the 2^{nd} term. Integrate by parts to get

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \frac{d\eta}{dx}\ x = \left[\frac{\partial f}{\partial y'}\eta \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta\ dx$$

$$= -\int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta\ dx$$

since $\eta(x_1) = \eta(x_2) = 0$

So

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right] \eta\ dx$$

We want the above integral to vanish for <u>any</u> choice of of $\eta(x)$. The only way this can happen is if

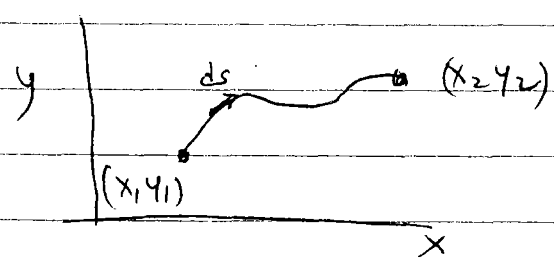
$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

This is <u>Euler's</u> equation – it gives a differential equation which ~~the~~ $y(x)$ must satisfy, if it is to give an extremal value of J. Solving this differential equation is how we find the desired function $y(x)$.

<u>Examples</u>

<u>(1) Shortest distance between two points</u>

What curve $y(x)$ gives the shortest distance from (x_1, y_1) to (x_2, y_2)?



infinitesimal arclength is

$$ds = \sqrt{dx^2 + dy^2}$$

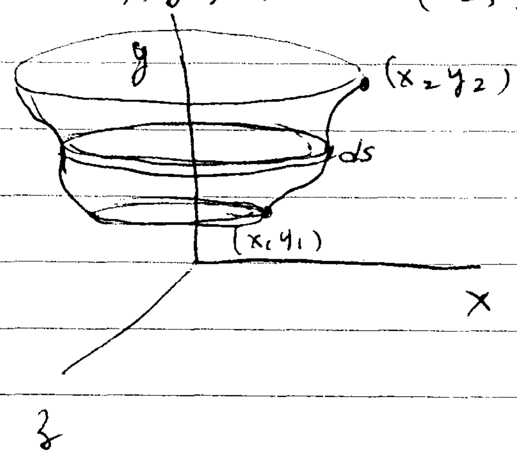
The total length of the curve is therefore

$$J = \int_1^2 ds = \int_1^2 \sqrt{dx^2 + dy^2} = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx$$

② Minimum surface of rotation

Consider the surface ~~upon~~ created by the rotation about the y-axis of the curve lying between $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$



the area of the circular strip of width $ds = \sqrt{dx^2 + dy^2}$ is

$$dS = 2\pi x\, ds$$
$$= 2\pi x \sqrt{dx^2 + dy^2}$$
$$= 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

The area of the surface is therefore

$$S = 2\pi \int_{x_1}^{x_2} x \sqrt{1 + (y')^2}\, dx$$

the functional is $\quad f[y, y'; x] = x\sqrt{1 + (y')^2}$

$$\frac{\partial f}{\partial y} = 0 \quad , \quad \frac{\partial f}{\partial y'} = \frac{x y'}{\sqrt{1 + (y')^2}}$$

Euler's eqn is

$$0 - \frac{d}{dx}\left(\frac{x y'}{\sqrt{1 + (y')^2}} \right) = 0$$

$$\Rightarrow \quad \frac{x y'}{\sqrt{1 + (y')^2}} = a \qquad a \text{ constant}$$

$$\frac{x^2(y')^2}{1+(y')^2} = a^2 \Rightarrow x^2(y')^2 = a^2 + a^2(y')^2$$

$$\Rightarrow (y')^2(x^2-a^2) = a^2$$

$$\Rightarrow y' = \frac{dy}{dx} = \frac{a}{\sqrt{x^2-a^2}}$$

$$y = \int dx \frac{a}{\sqrt{x^2-a^2}} = a\cosh^{-1}\left(\frac{x}{a}\right) + b$$

or $x = a\cosh\left(\frac{y-b}{a}\right)$ this is equation of a "catenary" - the curve of a flexible cord hanging between two points of support.

a and b determined by $y(x_1) = y_1$, $y(x_2) = y_2$

This also gives solution for the surface area of a soap film suspended between two rings of radii x_1 and x_2 a height $y_2 - y_1$ apart.