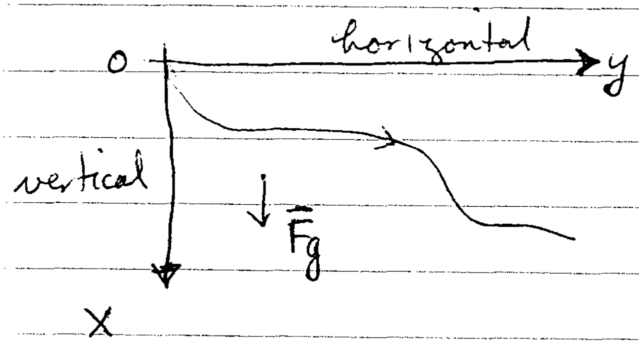
③ The roller coaster problem — the Brachistochrone problem

Find the path $y(x)$ for the car to reach the end in the shortest time. Assume track is frictionless, and initial $v=0$ (this is problem that stimulated the development of the calculus of variations by Bernoulli in 1696)

Energy is conserved $E = \frac{1}{2}mv^2 - mgx$

kinetic potential



note because of how we choose x axis as positive downwards

$U = -mgx$

If start at $x=y=0$, with $v=0$, then $E=0$

So

$$\frac{1}{2}mv^2 = mgx$$

$$v = \sqrt{2gx}$$

infinitesimal arclength is $ds = \sqrt{dx^2+dy^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

time $dt = \frac{ds}{v} = \frac{\sqrt{1+(y')^2}}{\sqrt{2gx}}\,dx = \left(\frac{1+(y')^2}{2gx}\right)^{1/2} dx$

So we want to minimize.

$$\int_0^{x_2} \left(\frac{1+(y')^2}{2gx}\right)^{1/2} dx$$

functional $f[y, y'; x] = \sqrt{\frac{1+(y')^2}{2gx}}$

$$\frac{\partial f}{\partial y} = 0 \quad , \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{(1+(y')^2)\,2gx}}$$

Euler

$$0 - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$\Rightarrow \quad \frac{y'}{\sqrt{(1+(y')^2)\,2gx}} = \text{const}$$

$$\Rightarrow \quad \frac{y'^2}{(1+y'^2)x} = \text{const} = \frac{1}{2a}$$

$$y'^2 = (1+y'^2)\frac{x}{2a} = \frac{x}{2a} + \frac{x}{2a}y'^2$$

$$y'^2 = \frac{x/2a}{1-x/2a} = \frac{x}{2a-x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \sqrt{\frac{x}{2a-x}} = \frac{x}{\sqrt{2ax-x^2}}$$

$$\Rightarrow \quad y = \int dx \, \frac{x}{\sqrt{2ax-x^2}}$$

Make the change of variables

$$x = a(1-\cos\theta)$$

$$dx = a\sin\theta\, d\theta$$

$$y = \int \frac{a(1-\cos\theta)\quad a\sin\theta\, d\theta}{\sqrt{2a^2(1-\cos\theta) - a^2(1-\cos\theta)^2}}$$

$$= \int \frac{a^2(1-\cos\theta)\sin\theta\, d\theta}{a\sqrt{2-2\cos\theta - 1 - \cos^2\theta + 2\cos\theta}}$$

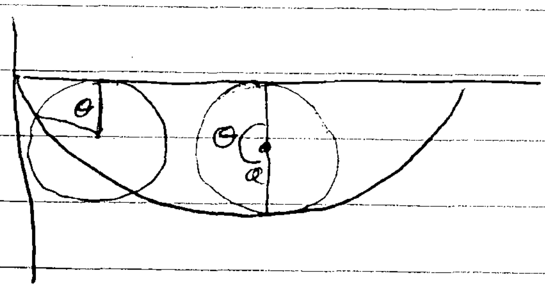
$$= \int \frac{a(1-\cos\theta)\sin\theta\, d\theta}{\sqrt{1-\cos^2\theta}}$$

$$y = \int a(1-\cos\theta)\, d\theta$$

const of integ $=0$ if $y(\theta=0)=0$

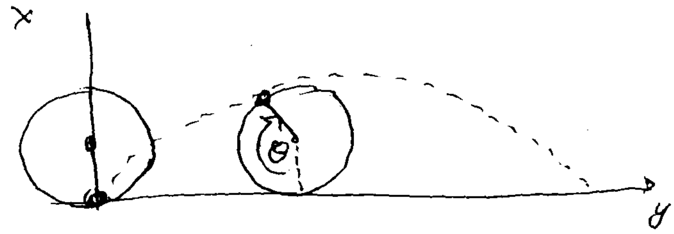
$$\left.\begin{aligned} y &= a(\theta - \sin\theta) \\ \text{where}\quad x &= a(1-\cos\theta) \end{aligned}\right\}$$

these are parametric equations of cycloid passing through origin



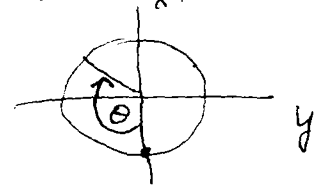
curve traced out by a point on the rim of a rolling wheel.

parametric curve of a cycloid



wheel of radius a

put origin of x', y' coordinates at center of wheel.



in x'-y' coordinates, the position of the point on the rim of the wheel, after it rotates by angle θ is given by

$$x' = -a\cos\theta$$
$$y' = -a\sin\theta$$

Now, the center of the wheel, as measured in the fixed x-y coordinate system is at

$$x = a$$
$$y = vt$$ where v is speed of center of wheel.

when the wheel has moved a distance vt, it has rotated through an angle $\theta = vt/a$

$\Rightarrow$ center of wheel is at $x = a$, $y = a\theta$.

$\rightarrow$ x-y coordinates of point on rim are

$$x = a + x' = a - a\cos\theta = \boxed{a(1-\cos\theta) = x}$$
$$y = a\theta + y' = a\theta - a\sin\theta = \boxed{a(\theta - \sin\theta) = y}$$

The time of travel on this curve is

$$t = \int \sqrt{\frac{1+(y')^2}{2gx}}\, dx = \int \sqrt{\frac{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}{2gx}}\, d\theta$$

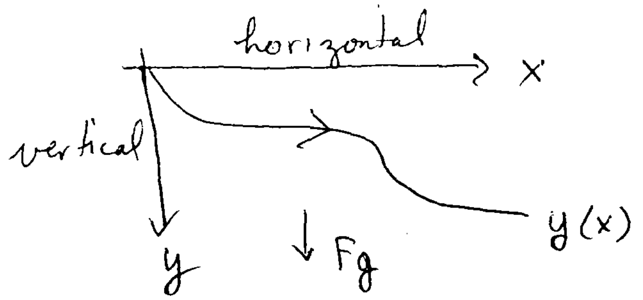
$$= \int \sqrt{\frac{a^2 \sin^2\theta + a^2(1-\cos\theta)^2}{2ga(1-\cos\theta)}}\, d\theta$$

$$= \int \sqrt{\frac{a^2(2-2\cos\theta)}{2ga(1-\cos\theta)}}\, d\theta$$

$$t = \sqrt{\frac{a}{g}}\, \theta$$

a is adjusted so curve passes through the desired point (x_2, y_2)

Note, we were able to solve this problem only because we chose our coordinate system in a convenient way. We could instead have chosen it as follows:



$$E = \frac{1}{2} m v^2 - m g y = 0$$

$$v = \sqrt{2 g y}$$

$$dt = \frac{ds}{v} = \frac{\sqrt{1 + (y')^2}}{\sqrt{2 g y}}\, dx$$

total time $\quad t = \displaystyle\int_{x_1}^{x_2} \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}\, dx$

functional $\quad f[y, y'; x] = \sqrt{\dfrac{1+y'^2}{2gy}}$

$$\frac{\partial f}{\partial y} = -\frac{1}{2y}\sqrt{\frac{1+y'^2}{2gy}}$$ ← this is no longer zero as it was before

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{(1+y'^2)\, 2gy}}$$

Eulers equ:

$$-\frac{1}{2y}\sqrt{\frac{1+y'^2}{2gy}} - \frac{d}{dx}\left(\frac{y'}{\sqrt{(1+y'^2)\, 2gy}}\right) = 0$$

Much harder to find the solution $y(x)$ to this equation than it was before.

Conclusion: If one can pick coordinates so that $\frac{\partial f}{\partial y} = 0$ this usually makes the problem easier.

Second form of Euler's equations

Note: ① $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx}$

Also ② $\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y'}\frac{dy'}{dx} + y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$

by chain rule

from ① we get $\frac{\partial f}{\partial y'}\frac{dy'}{dx} = \frac{df}{dx} - \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}y'$

Substitute into ② to get

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = \frac{df}{dx} - \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}y' + y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

Rearrange to get

$$\frac{\partial f}{\partial x} - \frac{df}{dx} + \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = -y'\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)$$

or

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = -y'\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)$$

Now if $y(x)$ is a solution of Euler's equation, the right hand side of the above vanishes. This implies that $y(x)$ must also be a solution of

$$\boxed{\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0}$$

This is known as the <u>second form</u> of Euler's equation.

This form is convenient for situations in which $\frac{\partial f}{\partial x} = 0$.

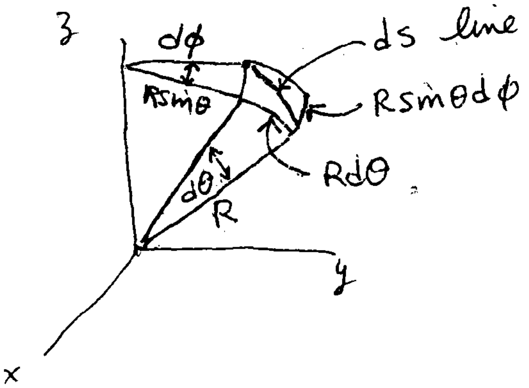
In this case we get $\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$

or $\quad f - y'\frac{\partial f}{\partial y'} = \text{constant} \qquad \left(\text{when } \frac{\partial f}{\partial x} = 0\right)$

<u>Example</u> : The <u>geodesic</u>

a geodesic line is the shortest distance between two points when the path is restricted to lie on a particular surface (for a plane surface, we know already that the geodesic is a straight line).

Consider instead the surface of a sphere of radius R.



$$ds = \sqrt{(R\,d\theta)^2 + (R\sin\theta\, d\phi)^2}$$

$$ds = R\sqrt{d\theta^2 + \sin^2\theta\, d\phi^2}$$

$$ds = R\sqrt{\left(\frac{d\theta}{d\phi}\right)^2 + \sin^2\theta}\; d\phi$$

length of line is

$$S = R\int_1^2 \sqrt{\left(\frac{d\theta}{d\phi}\right)^2 + \sin^2\theta}\; d\phi$$

functional is

$$f[\theta, \theta'; \phi] = \sqrt{(\theta')^2 + \sin^2\theta}$$

where $\theta' = \frac{d\theta}{d\phi}$

Here we have $\frac{\partial f}{\partial \phi} = 0$. We can therefore try the 2nd form of Eulers eqn

$$\Rightarrow \quad f - \theta' \frac{\partial f}{\partial \theta'} = \text{const}$$

where $\displaystyle \frac{\partial f}{\partial \theta'} = \frac{\theta'}{\sqrt{(\theta')^2 + \sin^2\theta}}$

$$\Rightarrow \quad \sqrt{(\theta')^2 + \sin^2\theta} - \frac{(\theta')^2}{\sqrt{(\theta')^2 + \sin^2\theta}} = \text{const} \equiv a$$

multiply through by $\sqrt{(\theta')^2 + \sin^2\theta}$

$$\Rightarrow \quad (\theta')^2 + \sin^2\theta - (\theta')^2 = a\sqrt{(\theta')^2 + \sin^2\theta}$$

$$\sin^2\theta = a\sqrt{(\theta')^2 + \sin^2\theta}$$

solve for θ' $\quad \Rightarrow \theta' = \dfrac{\sin\theta\sqrt{\sin^2\theta - a^2}}{a} = \dfrac{d\theta}{d\phi}$

$$\Rightarrow \quad d\phi = \frac{a\, d}{\sin\theta\sqrt{\sin^2\theta - a^2}} = \frac{a\csc^2\theta\, d\theta}{\sqrt{1 - a^2\csc^2\theta}}$$

$\csc\theta = \dfrac{1}{\sin\theta}$

solve by integration $\quad \displaystyle \int d\phi = \int \frac{a\csc^2\theta\, d\theta}{\sqrt{1 - a^2\csc^2\theta}}$

look up solution to integral

solution is

$$\phi = \arcsin\left(\frac{\cot\theta}{\beta}\right) + \alpha \qquad \text{where } \beta \equiv \frac{1-a^2}{a^2}$$

α is const of integration

$$\Rightarrow \beta \sin(\phi - \alpha) = \cot\theta$$

α and β determined by limits $\theta(\phi_1) = \theta_1$, $\theta(\phi_2) = \theta_2$

$$\beta \sin\phi \cos\alpha - \beta \cos\phi \sin\alpha = \frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow (\beta\cos\alpha)\sin\theta\sin\phi - (\beta\sin\alpha)\sin\theta\cos\phi = \cos\theta$$

or $$A\sin\theta\sin\phi - B\sin\theta\cos\phi = \cos\theta$$

where $A = \beta\cos\alpha$ and $B = \beta\sin\alpha$ are reparameterization of the two free constants, to be determined by $\theta(\phi_1)=\theta_1$, and $\theta(\phi_2)=\theta_2$.

If we multiply both sides by R we get

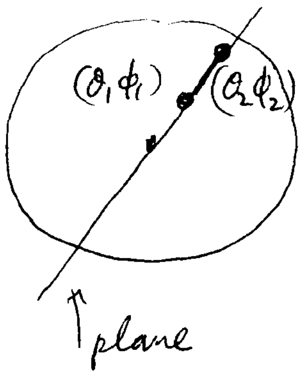
$$A(R\sin\theta\sin\phi) - B(R\sin\theta\cos\phi) = R\cos\theta$$

or $$\boxed{Ay - Bx = z}$$

where (x, y, z) are the Cartesian coordinates of the point at spherical coordinates (R, θ, ϕ)

this is the equation of a plane passing through the origin

Therefore, the geodesic on a sphere is the path connecting the two points that lies in a plane containing the two points and the center of the sphere (any three non-colinear points define a plane)



Such geodesic paths are called "great circles".

Note: Eulers method gives both the min and the max solution.

The min distance is the shortest distance along the great circle defined above.

The max distance is the longest distance along the same great circle.

airplanes making trans atlantic flights usually try to fligh on close to great circle routes.

Calculus of Variations – for functions with several dependent variables.

$f[y_1, y_1', y_2, y_2', \ldots, y_n, y_n'; x]$

where all the y_i are functions of x

The generalization of Euler's equation is now to a set of equations

$$\boxed{\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0 \qquad i = 1, 2, \ldots, n}$$

each equation comes from the fact that

$J = \int f[y_i, y_i'; x]\,dx$ must be extremal

with respect to variations in any one of the y_i