Calculus of Variations – for functions with several dependent variables.

$f[y_1, y_1', y_2, y_2', \ldots, y_n, y_n'; x]$

where all the y_i are functions of x

The generalization of Euler's equation is now to a set of equations

$$\boxed{\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0 \qquad i = 1, 2, \ldots, n}$$

each equation comes from the fact that

$J = \int f[y_i, y_i'; x]\, dx$ must be extremal

with respect to variations in any one of the y_i

Eulers equations with Constraints

Constraints:

<u>holonomic</u> constraints can be expressed in equ of form

$$g(y_1, y_2, \dots y_n; x) = 0$$

example: a particle confined to surface of a sphere obeys constraint

$$g(x,y,z) = x^2+y^2+z^2-R = 0$$

<u>non holonomic</u> constraints cannot be expressed in the above form

example: a particle confined to the <u>interior</u> of a sphere, $x^2+y^2+z^2-R^2 \leq 0$

In mechanics, Constraints are generally imposed by forces. the goal is to treat the constraint in a way where we do not need to deal directly with these forces.

For a problem with N degrees of freedom, subject to ~~and~~ m holonomic constraints, there are $s = N-m$ <u>independent</u> degrees of freedom.

We want to write equations of motion directly for these s degrees of freedom.

generalized coordinates

One approach is to use the constraint to reduce the set of N degrees of freedom + m constraints, to $s = N - m$ independent degrees of freedom q_i $(i = 1, 2, \ldots, s)$. If one rewrites the functional in terms of these q_i, one can then apply the Euler equ

$$\frac{\partial f}{\partial q_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial q_i'}\right) = 0$$

Lagrange multipliers - another approach

Consider two dependent variables $y(x)$ and $z(x)$ with the constraint $g(y, z; x) = 0$.

We want to minimize

$$J = \int dx\, f[y, y'; z, z'; x]$$

subject to the constraint $g = 0$.

~~The extremum of J is given by~~

$$\cancel{\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]}$$

Consider the variation about the minimizing $y(x), z(x)$

$$y_\alpha(x) = y(x) + \alpha \eta_1(x)$$

$$z_\alpha(x) = z(x) + \alpha \eta_2(x)$$

As before we can write

$$0 = \frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} dx \left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \eta_1 + \left(\frac{\partial f}{\partial z} - \frac{d}{dx} \frac{\partial f}{\partial z'} \right) \eta_2 \right]$$

however, now η_1 and η_2 are not independent of each other, but are constrained by

$$g(y + \alpha\eta_1, z + \alpha\eta_2 ; x) = 0$$

differentiating

$$\Rightarrow \quad \frac{dg}{d\alpha} = \frac{\partial g}{\partial y} \eta_1 + \frac{\partial g}{\partial z} \eta_2 = 0$$

Since above is always zero, we can multiply it by any arbitrary function $\lambda(x)$, and add it to the equation for $\frac{\partial J}{\partial \alpha}$

$$0 = \frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} dx \left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} + \lambda \frac{\partial g}{\partial y} \right) \eta_1 + \left(\frac{\partial f}{\partial z} - \frac{d}{dx} \frac{\partial f}{\partial z'} + \lambda \frac{\partial g}{\partial z} \right) \eta_2 \right]$$

Now since $\lambda(x)$ is arbitrary, let us choose it so that

$$\frac{\partial f}{\partial z} - \frac{d}{dx} \frac{\partial f}{\partial z'} + \lambda \frac{\partial g}{\partial z} = 0$$

ie: $$\lambda(x) = \left\{ \frac{d}{dx}\left(\frac{\partial f[y(x), y'(x), z(x), z'(x); x]}{\partial z'}\right) - \frac{\partial f[y(x), y'(x), z(x), z'(x); x]}{\partial z} \right\} \frac{1}{\left(\frac{\partial g[y(x), z(x); x]}{\partial z}\right)}$$

with this special choice for the function $\lambda(x)$, $\frac{\partial J}{\partial \alpha}$ now becomes

$$0 = \frac{\partial J}{\partial \alpha} = \int dx \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \lambda \frac{\partial g}{\partial y}\right]\eta_1$$

Since this must vanish for any choice of η_1, and η_1 is an independently chosen function (η_2 no longer appears in the above expression)

we conclude that

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \lambda\frac{\partial g}{\partial y} = 0$$

combining with our initial equation which defined $\lambda(x)$ we conclude

$$\left.\begin{aligned} &\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \lambda(x)\frac{\partial g}{\partial y} = 0 \\ &\frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right) + \lambda(x)\frac{\partial g}{\partial z} = 0 \\ &g(y, z; x) = 0 \end{aligned}\right\}$$

system of three equations for three unknown functions $y(x)$, $z(x)$, $\lambda(x)$

To generalize to N degrees of freedom y_i $(i=1,\dots,N)$ and m constraints $g_k(y_1,\dots,y_N;x)=0$ $(k=1,\dots,m)$ we get the set of equations

$N+m$ eqn for $N+k$ unknown functions

$$\left\{\begin{array}{ll} \dfrac{\partial f}{\partial y_i} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y_i'}\right) + \displaystyle\sum_{k=1}^{m} \lambda_k(x)\,\dfrac{\partial g_k}{\partial y_i} = 0 & i=1,\dots,N \\ g_k(y_1,\dots,y_N;x) = 0 & k=1,\dots,m \end{array}\right.$$

Solve for $y_1(x),\dots,y_N(x)$
and for $\lambda_1(x),\dots,\lambda_m(x)$

The functions $\lambda_k(x)$ are called the <u>Lagrange multipliers</u>

We will see that, in the context of classical mechanics, $\lambda_k(x)$ are related the forces that impose the constraints on the dynamical degrees of freedom

Calculus of Variations + Classical Mechanics

Hamilton's Principle + Lagrange's Eqn of Motion

Newton's 2nd law: $\vec{F} = \frac{d\vec{p}}{dt}$

kinetic energy $T = \frac{1}{2} m \dot{x}^2 = \frac{p^2}{2m}$

$$\Rightarrow \quad \frac{\partial T}{\partial \dot{x}} = p_x$$

and $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = \frac{dp_x}{dt} = F_x$$

If the total force $\vec{F} = -\vec{\nabla} U + \vec{F}^{NC}$

conservative parts of force — non-conservative parts of force

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = -\frac{\partial U}{\partial x} + F_x^{NC}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) + \frac{\partial U}{\partial x} = F_x^{NC}$$

Define the Lagrangian $\mathcal{L} = T - U$ = kinetic − potential

Since T is indep of $\vec{r}$, and U is indep of $\dot{\vec{r}}$

then
$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x}$$

$\Rightarrow$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = F_x^{NC}$$ similarly for y, z components

This looks just like Euler's equations with Lagrange multipliers if

$$F_x^{NC} = \sum_{k=1}^{m} \lambda_k(x) \frac{\partial g_k}{\partial y_i}$$

$\Rightarrow$ Lagrange ~~mult~~ multiplier is related to non conservative components of the force

Now suppose $\vec{F}^{NC} = 0$, ie all forces conservative then

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$ $\leftarrow$ Lagrange's Equ of motion

this is just Euler's equ for the minimization of

"Action" = $J = \int_{t_1}^{t_2} \mathcal{L}[\vec{r}(t), \dot{\vec{r}}(t); t]\, dt$ subject to conditions $\vec{r}(t_1) = \vec{r}_1$, $\vec{r}(t_2) = \vec{r}_2$

We thus arrive at Hamilton's Principle: ~~the~~

If all forces are conservative, then the trajectory $\vec{r}(t)$ of a particle which moves from position $\vec{r}(t_1) = \vec{r}_1$ to position $\vec{r}(t_2) = \vec{r}_2$, is the curve that ~~minimizes~~ minimizes the time integral of the Lagrangian.

$\Rightarrow$ "principle of least action"

(*) Follows from Hamilton's principle — trajector in any coordinates is the one that minimizes $\int_{t_1}^{t_2} \mathcal{L}\, dt$

The Lagrangian $\mathcal{L}$ is a <u>scalar</u> function of the degrees of freedom. Its convenience lies in the fact that these degrees of freedom need <u>not</u> be the usual rectangular coordinates x, y, z, but instead can be taken as <u>any</u> generalized coordinates that specify the state of the system.(*) One denotes the generalized coordinates $\{q_i\}$ and their generalized velocities $\{\dot{q}_i\}$. For $\mathcal{L}[q_i, \dot{q}_i; t]$ the equations of motion for the q_i are then

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

for unconstrained independent coords q_i

<u>examples</u>: ① motion of a free particle.

coordinates x, y, z

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \qquad U = 0$$

$$\mathcal{L} = \tfrac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0 \quad , \quad \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = m\dot{x}_i$$

Lagrange eqn: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \frac{\partial \mathcal{L}}{\partial x_i} = 0 \Rightarrow \frac{d}{dt}(m\dot{x}_i) = 0$

$$\Rightarrow m\dot{x}_i = \text{const}$$

<u>momentum is conserved</u>

② harmonic oscillator:

coordinate: displacement x

$$T = \frac{1}{2} m \dot{x}^2 \qquad U = \frac{1}{2} k x^2 \qquad \mathcal{L} = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

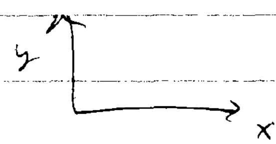
$$\frac{\partial \mathcal{L}}{\partial x} = -kx \ , \quad \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = 0 \Rightarrow \frac{d}{dt}(m\dot{x}) + kx = 0$$

$$m\ddot{x} + kx = 0$$

we get familiar harmonic osc. equ: $\ddot{x} + \frac{k}{m} x = 0$

③ motion in gravitational field



$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \qquad U = +mgy$$

$$\mathcal{L} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy$$

x coord: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}(m\dot{x}) - 0 = \frac{d}{dt}(m\dot{x}) = 0 \Rightarrow \ddot{x} = 0$

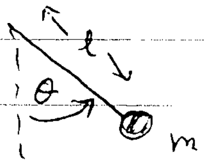
$$\Rightarrow \dot{x} = \text{const}$$

y coord: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}(m\dot{y}) + mg = 0$

$$\Rightarrow \ddot{y} = -g$$

In above 3 examples, Lagrange method reproduces Newtons laws
results from

(4) pendulum



coord: θ ← first example of generalized coordinate!

$$T = \frac{1}{2}m(l\dot{\theta})^2 \qquad U = -mgl\cos\theta$$

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = \frac{d}{dt}(ml^2\dot{\theta}) + mgl\sin\theta$$

$$= ml^2\ddot{\theta} + mgl\sin\theta = 0$$

$$\Rightarrow \ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

for small θ, $\sin\theta \simeq \theta \Rightarrow \ddot{\theta} + \frac{g}{l}\theta = 0$

harmonic oscillator with freq $\omega_0 = \sqrt{\frac{g}{l}}$

we got our result from simple scalar equations involving only kinetic & potential energy. Compare to method using Newton's equ which are vector equ using forces, or angular momentum and torque.