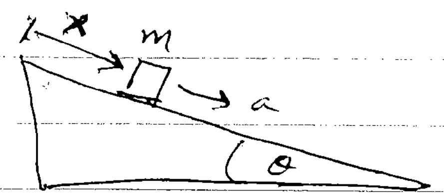
⑤ block on inclined plane



coordinate x = dist from top of incline

$T = \frac{1}{2} m \dot{x}^2 \qquad U = -mg\, x \sin\theta$

(if take origin at $x=0$, then height $h = -x\sin\theta$)

$\mathcal{L} = \frac{1}{2} m \dot{x}^2 + mg\, x \sin\theta$

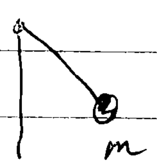
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}(m\dot{x}) - mg\sin\theta = 0$$

$$\Rightarrow \ddot{x} = g\sin\theta$$ acceleration down the plane

Same result as Newton, but did not need to worry about normal force! Did not need to project $\vec{F}_g$ into parallel and perpendicular components!

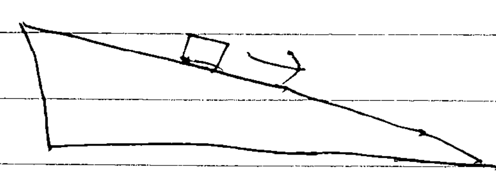
<u>Note:</u> In the last two examples we saw that the Lagrangian approach can be very helpful when there is an implied constraint.

④ pendulum



constraint is radial dist = const

⑤ block on plane

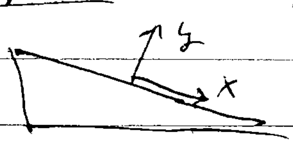


constraint is - block stays on surface of plane

In both cases we can think that there are really two degrees of freedom, and one constraint

<u>pendulum</u>: degrees of freedom = θ, r
constraint $r = l$ constant

<u>inclined plane</u>: degrees of freedom = x, y
constraint $y = 0$



We chose our generalized coordinates so that the constraint eliminated one coord, and the Lagrangian became a function of just the one remaining independent coordinate.

~~More generally, if there are N degrees of freedom and m holonomic constraints between them, then there are s = N−m independent degrees of freedom. One can always in principle use the constraints to rewrite ~~eliminate~~ m degrees of freedom, in terms of the others~~

More generally, if there are N degrees of freedom, and m holonomic constraints, then there are $s = N - m$ <u>independent</u> degrees of freedom. We can then choose s independent generalized coordinates q_i, $i = 1, \ldots, s$, such that we can write all the N degrees of freedom in terms of these ie $x_k = x_k(q_1, \dots, q_s) \qquad k = 1, \dots, N$

and $\dot{x}_k$ in terms of the $\dot{q}_i$

and then express the Lagrangian in terms of
$\mathcal{L}[q_i, \dot{q}_i; t]$
the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

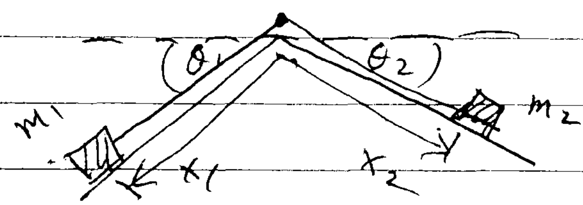
then give s ~~ind~~ equations for the s independent degrees of freedom.

The trick is to choose the q_i to make the resulting Lagrange eqns as simple as possible.

more examples

two masses tied together on a frictionless inclined plane

(6)



constraint: $x_1 + x_2 = l$ length of rope

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$U = -m_1 g x_1 \sin\theta_1 - m_2 g x_2 \sin\theta_2$$

since height $h_1 = -x_1 \sin\theta_1$, $h_2 = -x_2 \sin\theta_2$

use the constraint to eliminate x_2 in terms of x_1, then write $\mathcal{L}$ in terms of x_1

$$x_2 = \ell - x_1 \quad \Rightarrow \quad \dot{x}_2 = -\dot{x}_1$$

$$T = \tfrac{1}{2}(m_1 + m_2)\dot{x}_1^2$$

$$\begin{aligned} U &= -m_1 g x_1 \sin\theta_1 - m_2 g (\ell - x_1)\sin\theta_2 \\ &= -g(m_1\sin\theta_1 - m_2\sin\theta_2)x_1 - m_2 g \ell \sin\theta_2 \end{aligned}$$

a constant

(Note: adding a constant to $\mathcal{L}$ does not change the Lagrange equations of motion)

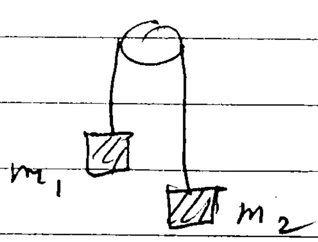
$$\mathcal{L} = \tfrac{1}{2}(m_1 + m_2)\dot{x}_1^2 + g(m_1\sin\theta_1 - m_2\sin\theta_2)x_1 - m_2 g \ell \sin\theta_2$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right) - \frac{\partial \mathcal{L}}{\partial x_1} = \frac{d}{dt}\left((m_1+m_2)\dot{x}_1\right) - g(m_1\sin\theta_1 - m_2\sin\theta_2) = 0$$

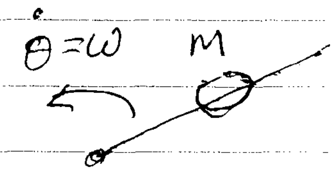
$$\Rightarrow \quad \ddot{x}_1 = g\,\frac{(m_1\sin\theta_1 - m_2\sin\theta_2)}{m_1 + m_2}$$

acceleration is zero if $m_1\sin\theta_1 = m_2\sin\theta_2$

<u>Atwoods machine</u> – $\theta_1 = \theta_2 = \frac{\pi}{2}$



$$\ddot{x}_1 = g\,\frac{(m_1 - m_2)}{m_1 + m_2}$$

(7) 

sliding bead of mass m on a rod that rotates with angular speed ω.

We did this earlier when we discussed velocity + acceleration in polar coordinates

coordinates: r, θ constraint $\dot{\theta} = \omega$ fixed

$$T = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m (r\dot{\theta})^2 \qquad U = 0$$

$$\mathcal{L} = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) = \frac{1}{2} m (\dot{r}^2 + r^2 \omega^2)$$

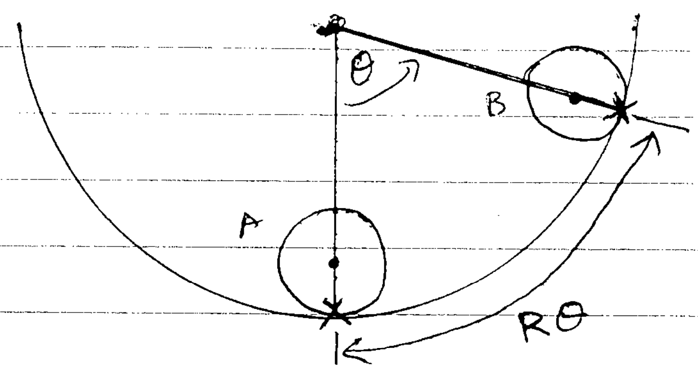
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}(m\dot{r}) - m r \omega^2 = 0$$

$$\ddot{r} = \omega^2 r$$

$$r(t) = r(0) e^{\omega t}$$

Same solution as found earlier

(8) prob 7-3



sphere of radius ρ rolling on cylinder of Radius R

Suppose that in rolling from position A to B, the point "x" on the sphere that is in contact with the cylinder at A, is once again in contact with the cylinder at B.

$\Rightarrow$ distance rolled $R\theta = 2\pi\rho$
in this process, the sphere has rotated through an angle $(2\pi - \theta) = \phi$

In general, if distance rolled is some fraction s of $2\pi\rho$, ie $R\theta = 2\pi s\rho$, then the sphere has rotated through $(2\pi s - \theta) = \phi \Rightarrow 2\pi s = \phi + \theta$

$$\Rightarrow \boxed{R\theta = (\theta + \phi)\rho}$$

constraint relating angle θ to angle ϕ of rotation of sphere about its center

We will choose θ as the generalized coordinate in which to do the problem.

$$\frac{(R-\rho)\theta}{\rho} = \phi \Leftarrow \text{constraint}$$

$$T = \frac{1}{2} m (R-\rho)^2 \dot{\theta}^2 + \frac{1}{2} I \dot{\phi}^2$$

center of mass KE   rotational KE

substitute for $\dot{\phi}$

$$T = \frac{1}{2} m (R-\rho)^2 \dot{\theta}^2 + \frac{1}{2}\frac{2}{5} m\rho^2 \frac{(R-\rho)^2}{\rho^2} \dot{\theta}^2$$

$$T = \frac{1}{2}\frac{7}{5} m (R-\rho)^2 \dot{\theta}^2$$

If place origin at the center of the cylinder, the potential energy of gravity is

$$U = -mg(R-\rho)\cos\theta$$

$$\Rightarrow \mathcal{L} = \frac{1}{2}\frac{7}{5} m (R-\rho)^2 \dot{\theta}^2 + mg(R-\rho)\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = \frac{7}{5} m (R-\rho)^2 \ddot{\theta} + mg(R-\rho)\sin\theta = 0$$

$$\Rightarrow \boxed{\ddot{\theta} + \frac{5g}{7(R-\rho)}\sin\theta = 0} \simeq \ddot{\theta} + \frac{5}{7}\frac{g}{(R-\rho)}\theta = 0$$

for small oscillations

same as a pendulum with angular freq

$$\omega = \sqrt{\frac{5}{7}\frac{g}{R-\rho}}$$

oscillates at freq $\omega = \sqrt{\dfrac{g}{R-\rho + \dfrac{I}{m(R-\rho)}\left(\dfrac{R}{\rho}\right)^2}}$

For a uniform sphere, $I = \frac{2}{5} m \rho^2$

$$\omega = \sqrt{\frac{g}{R-\rho + \frac{2}{5}\frac{R^2}{(R-\rho)}}}$$

if $\rho \ll R$ then

$$\omega \simeq \sqrt{\frac{g}{R + \frac{2}{5}R}} = \sqrt{\frac{5}{7}\frac{g}{R}}$$

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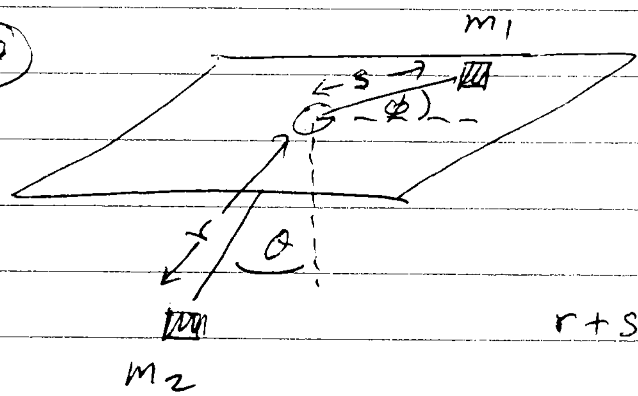


Connected mass, one on plane, other hanging

$r + s = \ell$ constraint

$$T = \tfrac{1}{2} m_1 \left(\dot{s}^2 + (s\dot{\phi})^2\right) + \tfrac{1}{2} m_2 \left(\dot{r}^2 + (r\dot{\theta})^2\right)$$

$$U = -m_2 g r \cos\theta$$

use $s = \ell - r$

$$\mathcal{L} = T - U = \tfrac{1}{2} m_1 \left(\dot{r}^2 + (\ell - r)^2 \dot{\phi}^2\right) + \tfrac{1}{2} m_2 \left(\dot{r}^2 + r^2 \dot{\theta}^2\right)$$

$$+ m_2 g r \cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot r} = m_1 \dot r + m_2 \dot r = (m_1+m_2)\dot r$$

$$\frac{\partial \mathcal{L}}{\partial r} = -m_1(\ell - r)\dot\phi^2 + m_2 r \dot\theta^2 + m_2 g \cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot\phi} = m_1(\ell-r)^2\dot\phi \qquad \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot\theta} = m_2 r^2 \dot\theta \qquad \frac{\partial \mathcal{L}}{\partial \theta} = -m_2 g r \sin\theta$$

<u>Lagrange eqns</u>: $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\phi}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$

$\Rightarrow m_1 \frac{d}{dt}\left[(\ell-r)^2\dot\phi\right] = 0$ $\Rightarrow$ conservation of $\hat{z}$ component of angular momentum

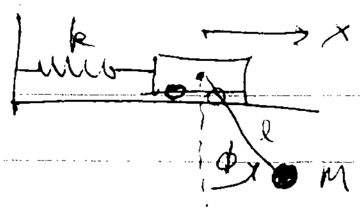
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \Rightarrow \frac{d}{dt}\left(m_2 r^2 \dot\theta\right) = -m_2 g r \sin\theta$$

change of angular momentum in vertical plane = torque from gravity

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot r}\right) - \frac{\partial \mathcal{L}}{\partial r} = 0 \Rightarrow$$

$$\cancel{\frac{d}{dt}\left(-m_1(\ell-r)\dot\phi^2 + m_2 r \dot\theta^2\right.}$$

$$(m_1+m_2)\ddot r + m_1(\ell-r)\dot\phi^2 - m_2 r\dot\theta^2 - m_2 g\cos\theta = 0$$

10 

pendulum of mass m attached to massless cart on spring.

coordinates x, ϕ

$T = \frac{1}{2} m v^2$ where $\vec{v}$ is velocity of mass m.

Choose coordinates x', y' centered on cart. Then mass is at

$$\left.\begin{array}{l} x' = l \sin\phi \\ y' = -l\cos\phi \end{array}\right\} \Rightarrow \left.\begin{array}{l} \dot{x}' = l\cos\phi\, \dot{\phi} \\ \dot{y}' = l\sin\phi\, \dot{\phi} \end{array}\right\}$$ velocity of mass relative to center of cart

to get velocity of mass in lab frame, we need addition of velocities

$$v_x = \dot{x} + \dot{x}' = \dot{x} + l\dot{\phi}\cos\phi$$

$$v_y = \dot{y}' = l\dot{\phi}\sin\phi$$

$$T = \frac{1}{2}m\left((\dot{x} + l\dot{\phi}\cos\phi)^2 + (l\dot{\phi}\sin\phi)^2 \right)$$

$$= \frac{1}{2}m\left(\dot{x}^2 + l^2\dot{\phi}^2 + 2\dot{x}l\dot{\phi}\cos\phi \right)$$

Potential energy $U = \frac{1}{2}kx^2 + mgy$

$$= \frac{1}{2}kx^2 - mg\,l\cos\phi$$

$$\mathcal{L} = T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}ml^2\dot{\phi}^2 + m\dot{x}l\dot{\phi}\cos\phi - \frac{1}{2}kx^2 + mgl\cos\phi$$

Lagrange equ for x:

$$\frac{\partial \mathcal{L}}{\partial x} = -kx \quad , \quad \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} + ml\dot{\phi}\cos\phi$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = m\ddot{x} + ml\ddot{\phi}\cos\phi - ml\dot{\phi}^2\sin\phi$$

$$\Rightarrow \boxed{m\ddot{x} + ml\ddot{\phi}\cos\phi - ml\dot{\phi}^2\sin\phi + kx = 0} \quad (1)$$

Lagrange equ for ϕ:

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m\dot{x}l\dot{\phi}\sin\phi - mgl\sin\phi$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = ml^2\dot{\phi} + m\dot{x}l\cos\phi$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = ml^2\ddot{\phi} + m\ddot{x}l\cos\phi - m\dot{x}l\dot{\phi}\sin\phi$$

$$\Rightarrow ml^2\ddot{\phi} + m\ddot{x}l\cos\phi - m\dot{x}l\dot{\phi}\sin\phi + m\dot{x}l\dot{\phi}\sin\phi + mgl\sin\phi = 0$$

$$\Rightarrow \boxed{ml^2\ddot{\phi} + m\ddot{x}l\cos\phi + mgl\sin\phi = 0} \quad (2)$$

need to solve above two differential equation for $x(t)$ and $\phi(t)$.

If small oscillations, then approx $\sin\phi \simeq \phi$, $\cos\phi \simeq 1$

$$\Rightarrow \begin{cases} m\ddot{x} + ml\ddot{\phi} - ml\dot{\phi}^2\phi + kx = 0 \\ m\ddot{x} + ml\ddot{\phi} + mg\phi = 0 \end{cases} \quad \leftarrow \text{divided (2) by } l$$

subtract

$$mg\phi + \underbrace{ml\dot{\phi}^2\phi}_{\text{high order in small quantities, so ignore}} + kx = 0$$

$\Rightarrow \quad mg\phi + kx = 0$

$\Rightarrow \quad x = \frac{-mg\phi}{k}$

~~inser~~ substitute into $\quad m\ddot{x} + m\ell\ddot{\phi} + mg\phi = 0 \quad$ to get

$$-\frac{m^2 g}{k}\ddot{\phi} + m\ell\ddot{\phi} + mg\phi = 0$$

$$\Rightarrow \quad \ddot{\phi}\left(1 + \frac{mg}{k\ell}\right) + \frac{g}{\ell}\phi = 0$$

$$\ddot{\phi} + \frac{\left(\frac{g}{\ell}\right)}{1 + \left(\frac{g}{\ell}\right)\left(\frac{m}{k}\right)}\phi = 0$$

pendulum oscillates with freq

$$\omega = \sqrt{\frac{g/\ell}{1 + (g/\ell)(m/k)}} = \sqrt{\frac{g}{\ell + gm/k}}$$

If the cart is fixed, this is equivalent to taking limit of infinitely stiff spring $k \to \infty$. In this case we recover ordinary pendulum result $\omega = \sqrt{g/\ell}$