Lagrange's Equations with Constraints

In previous examples we used constraints to eliminate degrees of freedom, and thus to write the Lagrangian in terms of only <u>independent</u> degrees of freedom.

Here we will keep all the degrees of freedom and introduce the constraints by the method of Lagrange multipliers. We will see that the Lagrange multiplier is related to the forces that imposes the constraints

we saw for Euler's equ with constraints

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \sum_{k=1}^{m} \lambda_k(t) \frac{\partial g_k}{\partial q_i} = 0 \qquad i = 1, \ldots, N$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \sum_{k=1}^{m} \lambda_k(t) \frac{\partial g_k}{\partial q_i}$$

q_i are the generalized coordinates of the problem

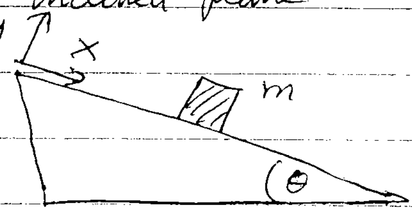
when q_i is an ordinary rectangular spatial coordinate, then $\sum_k \lambda_k(t) \frac{\partial g_k}{\partial q_i}$ is a force — as we saw in our derivation of Lagrange's equ from Newton's 2^{nd} law. We can see that this is dimensionally correct since $\mathcal{L}$ has units of energy, and if q_i has units of length, then $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i}$ has units of energy/length = units of force.

If q_i is not a length, then $\sum_{k=1}^{m} \lambda_k(t) \frac{\partial g_k}{\partial q_i}$ is called a "generalized force".
For example, if q_i is an angle, then $\sum_{k=1}^{n} \lambda_k(t) \frac{\partial g_k}{\partial q_i}$ has units of torque. Generalized force for q_i is denoted "Q_i".

$Q_i \delta q_i$ is work done by generalized force as coordinate changes by δq_i

Examples

inclined plane

② 

block stays on plane.

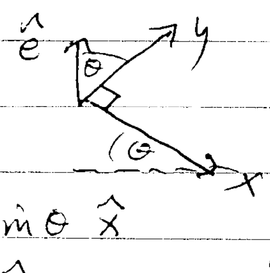
constraint $g(x,y) = y = 0$

Instead of using this constraint to eliminate y from Lagrangian $\mathcal{L}$ we keep both x and y and use Lagrange multiplier to handle constraint.

$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$

$U = -mg(\sin\theta\, x - \cos\theta\, y)$

follows since vertical direction $\hat{e}$ in x-y coordinates is



$\hat{e} = \cos\theta\, \hat{y} - \sin\theta\, \hat{x}$

So height is $\vec{r} \cdot \hat{e} = -x\sin\theta + y\cos\theta$

$$\mathcal{L} = T - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mg(x\sin\theta - y\cos\theta)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = \lambda \frac{\partial g}{\partial x}$$ $g(x,y) = y = 0$

since $\frac{\partial g}{\partial x} = 0$

$$\Rightarrow \quad m\ddot{x} - mg\sin\theta = 0 \qquad (1)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial \mathcal{L}}{\partial y} = \lambda \frac{\partial g}{\partial y}$$

since $\frac{\partial g}{\partial y} = 1$

$$\Rightarrow \quad m\ddot{y} + mg\cos\theta = \lambda \qquad (2)$$

(1) $m\ddot{x} = mg\sin\theta \Rightarrow \ddot{x} = g\sin\theta$ accel down plane as before

(2) $m\ddot{y} + mg\cos\theta = \lambda$

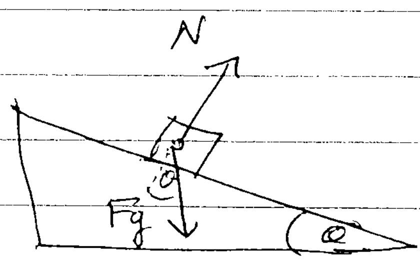
use constraint $y = 0 \Rightarrow \ddot{y} = 0$

so (2) $\Rightarrow mg\cos\theta = \lambda$

Forces of constraint are – give the forces not included in P.E. U

along x: $F_x = \lambda \frac{\partial g}{\partial x} = 0$

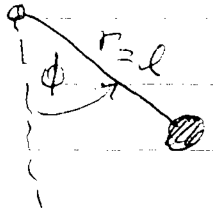
along y: $F_y = \lambda \frac{\partial g}{\partial y} = \lambda = mg\cos\theta$



this is just the normal force which is in the y direction!

$N = mg\cos\theta$

② pendulum



constraint of fixed length

$$g(r,\phi) = r - \ell = 0$$

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2)$$

$$U = -mgr\cos\phi$$

$$\mathcal{L} = T - U = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\phi}^2) + mgr\cos\phi$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = \lambda \frac{\partial g}{\partial \phi} \qquad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m r^3 \dot{\phi}$$

$$\Rightarrow \quad m r^2 \ddot{\phi} + 2 m \dot{r} \dot{\phi} + mgr\sin\phi = 0 \quad ① \text{ since } \frac{\partial g}{\partial \phi} = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = \lambda \frac{\partial g}{\partial r}$$

$$\Rightarrow m\ddot{r} - mr\dot{\phi}^2 - mg\cos\phi = \lambda \quad ② \text{ since } \frac{\partial g}{\partial r} = 1$$

constraint $r = \ell \quad \Rightarrow \dot{r} = 0, \ \ddot{r} = 0$

substitute into ① $\Rightarrow m\ell^2\ddot{\phi} + mg\ell\sin\phi = 0$

$\Rightarrow \ddot{\phi} + \frac{g}{\ell}\sin\phi = 0$ as found before

substitute into ②

$$\Rightarrow -m\ell\dot{\phi}^2 - mg\cos\phi = \lambda$$

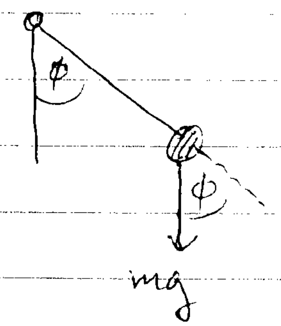
generalized forces of constraint

along ϕ: $\quad Q_\phi = \lambda \frac{\partial g}{\partial \phi} = 0 \quad$ = torque

along r: $\quad Q_r = \lambda \frac{\partial g}{\partial r} = \lambda \quad$ = radial force

$$= -mg\cos\phi - m\ell\dot{\phi}^2$$

Note: The force we find above is just what we expect:



For the circular motion, the radial component of acceleration is just the centripetal acceleration

$a_r = -l\dot{\phi}^2$

So $-ml\dot{\phi}^2$ is the net force in radial direction F_r.

$F_r = mg\cos\phi - T$ where T is tension in rope of length l

$\Rightarrow F_r = mg\cos\phi - T = -ml\dot{\phi}^2$

$\Rightarrow$ force of constraint $-T = -ml\dot{\phi}^2 - mg\cos\phi = \lambda$

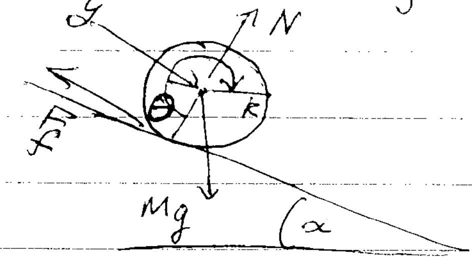
(since force of constraint computed is always in the direction of increasing values of the coordinate, we have that the force of constraint $\lambda \frac{\partial g}{\partial r}$ is $-T$ rather than T)

Force of constraint is always that part of the total force that has not been included in the potential energy U that entered the Lagrangian $\mathcal{L}$

In this case, this is just the tension T

More interesting examples

③ Disk rolling, without slipping, down inclined plane



y gives displ of center of disk from top of plane
θ gives angle of rotation of disk

First, to compare the methods, we can solve this problem the old Newtonian way.

force balance along y - parallel to surface:

① $M\ddot{y} = Mg\sin\alpha - F_f$ F_f is friction

force balance perpendicular to surface:

② $0 = -Mg\cos\alpha + N$ N is normal force

torque about center of disk

③ $F_f R = I\ddot{\theta}$ I is moment of inertia

no slipping constraint:

④ $y = R\theta$

④ $\Rightarrow \ddot{y} = R\ddot{\theta}$

substitute into ③ to get $F_f R = \frac{I}{R}\ddot{y}$

$\Rightarrow F_f = \frac{I}{R^2}\ddot{y}$

- :

substitute in ① to get

$$M\ddot{y} = Mg\sin\alpha - \frac{I}{R^2}\ddot{y}$$

$$\left(M + \frac{I}{R^2}\right)\ddot{y} = Mg\sin\alpha$$

$$\ddot{y} = \frac{Mg\sin\alpha}{M + \frac{I}{R^2}} = \frac{g\sin\alpha}{\left(1 + \frac{I}{MR^2}\right)}$$

for a solid circular disk, $I = \frac{1}{2}MR^2 \Rightarrow \ddot{y} = \frac{2}{3}g\sin\alpha$

frictional force $F_f = \frac{I}{R^2}\ddot{y} = \frac{Mg\sin\alpha}{1 + \frac{MR^2}{I}} = \frac{1}{3}Mg\sin\alpha$

<u>Now solve Lagrange's way</u>

$$T = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2$$

$U = -mgy\sin\alpha$ (origin is at top of incline)

Constraint $g(y,\theta) = y - R\theta = 0$

$$\mathcal{L} = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2 + Mgy\sin\alpha$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{y}}\right) - \frac{\partial\mathcal{L}}{\partial y} = \lambda\frac{\partial g}{\partial y}$$

$\Rightarrow M\ddot{y} - Mg\sin\alpha = \lambda$ as $\frac{\partial g}{\partial y} = 1$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = \lambda \frac{\partial g}{\partial \theta}$$

$$\Rightarrow \quad I\ddot\theta = -\lambda R \qquad \text{as } \frac{\partial g}{\partial \theta} = -R$$

$$\left.\begin{array}{ll} (1) & M\ddot y - Mg\sin\alpha = \lambda \\ (2) & I\ddot\theta = -\lambda R \\ (3) & y = R\theta \end{array}\right\} \text{ three equations to solve for } y, \theta, \lambda$$

~~Substitute~~ (3) $\Rightarrow \ddot y = R\ddot\theta$

substitute into (2) to get

$$I\frac{\ddot y}{R} = -\lambda R \quad \Rightarrow \quad \lambda = -\frac{I\ddot y}{R^2}$$

Substitute into (1) to get

$$M\ddot y - Mg\sin\alpha = -\frac{I\ddot y}{R^2}$$

$$\Rightarrow \left(M + \frac{I}{R^2}\right)\ddot y = Mg\sin\alpha$$

$$\ddot y = \frac{g\sin\alpha}{1 + \frac{I}{MR^2}}$$ as in Newtonian solution

So $$\lambda = -\frac{I\ddot y}{R^2} = -\frac{I}{R^2}\,\frac{g\sin\alpha}{1+\frac{I}{MR^2}} = \frac{-g\sin\alpha}{\frac{R^2}{I}+\frac{1}{M}}$$

$$= -\frac{Mg\sin\alpha}{1+\frac{MR^2}{I}}$$

- :

$\Rightarrow$ generalized forces of constraint

in y direction: $Q_y = F_f = \lambda \frac{\partial g}{\partial y} = \lambda = \frac{-Mg\sin\alpha}{1+\frac{MR^2}{I}}$

same as in Newtonian solution
— the minus sign is because F_f points in negative y direction

in θ direction: torque $Q_\theta = \tau = \lambda \frac{\partial g}{\partial \theta} = -\lambda R$

$$\tau = -F_f R = R|F_f|$$

this is just the torque due to the frictional force

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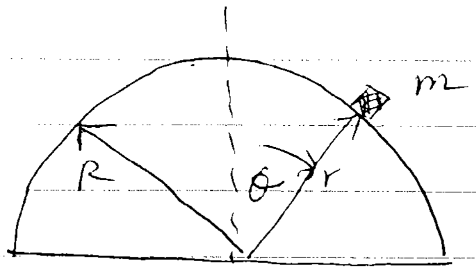
④ Mass sliding on frictionless hemispherical surface



coordinates r, θ

constraint $g(r,\theta) = r - R = 0$

(many similarities to pendulum prob)

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$$

$$U = mgr\cos\theta$$

$$\mathcal{L} = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = \lambda \frac{\partial g}{\partial r}$$

$$\Rightarrow \quad m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta = \lambda \quad ① \quad \text{since } \frac{\partial g}{\partial r} = 1$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = \lambda \frac{\partial g}{\partial \theta}$$

$$\Rightarrow mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mgr\sin\theta = 0 \quad \text{since } \frac{\partial g}{\partial \theta} = 0$$

constraint $r = R$ ③

from ③, $\ddot{r} = 0$, $\dot{r} = 0$

substitute into ① $\Rightarrow$ $-mR\dot{\theta}^2 + mg\cos\theta = \lambda$

② $\Rightarrow$ $mR^2\ddot{\theta} - mgR\sin\theta = 0$

$$\Rightarrow \quad \ddot{\theta} = \frac{g}{R} \sin\theta$$

we can integrate this using the following trick

$$\ddot{\theta} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$$

$$\dot{\theta}\frac{d\dot{\theta}}{d\theta} = \frac{g}{R}\sin\theta \Rightarrow \int \dot{\theta}\, d\dot{\theta} = \int \frac{g}{R}\sin\theta\, d\theta$$

$$\frac{1}{2}\dot{\theta}^2 = \frac{g}{R}(1-\cos\theta)$$

constant of integration set by
assuming $\dot{\theta}=0$ when $\theta=0$
(mass starts at rest on top of hemisphere)

$$\dot{\theta}^2 = \frac{2g}{R}(1-\cos\theta)$$

$$\Rightarrow \lambda = mg\cos\theta - mR\dot{\theta}^2 = mg\cos\theta - mR\frac{2g}{R}(1-\cos\theta)$$

$$= mg(\cos\theta - 2 + 2\cos\theta) = mg(3\cos\theta - 2)$$

The force of constraint in the radial direction is just the normal force from the surface. We get

$$N = \lambda\frac{\partial g}{\partial r} = \lambda = mg(3\cos\theta - 2)$$

We can now find out something interesting!

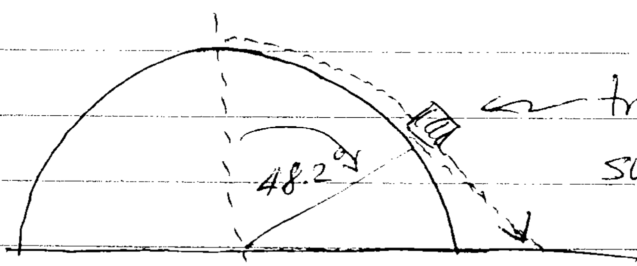
By definition, the normal force must be positive
– ie pointing outward from surface

But the expression we found

$$N = mg\,(3\cos\theta - 2)$$

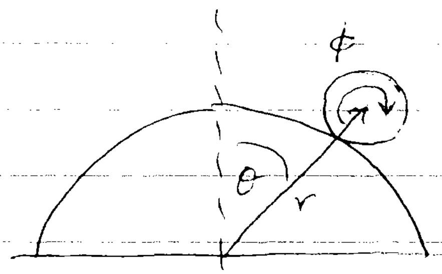
will become negative when $\cos\theta = \frac{2}{3}$,
ie when $\theta = 48.2^\circ$

What this tells us is that for $\theta > 48.2^\circ$
the normal force cannot provide what is
needed to maintain the constraint
$\Rightarrow$ the mass will fly off the surface!



← trajector of mass leaves
surface of hemisphere
when $\theta > 48.2^\circ$

⑤ Rolling disk on hemispherical surface



disk has radius a
hemisphere has radius R
coordinates: θ, r give center of mass of disk
ϕ gives angle of rotation of disk

constraint ①: disk rolls on surface

$$g_1(r,\theta,\phi) = r - (R+a) = 0$$

constraint ②: disk rolls without slipping

$$g_2(r,\theta,\phi) = a(\phi-\theta) - R\theta = 0$$

(similar to problem of ball rolling inside cylinder)

$$T = \tfrac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + \tfrac{1}{2} I \dot{\phi}^2$$

$$U = mgr\cos\theta$$

$$\mathcal{L} = \tfrac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + \tfrac{1}{2} I\dot{\phi}^2 - mgr\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = \lambda_1 \frac{\partial g_1}{\partial r} + \lambda_2 \frac{\partial g_2}{\partial r}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}, \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = m\ddot{r}, \quad \frac{\partial \mathcal{L}}{\partial r} = mr\dot{\theta}^2 - mg\cos\theta$$

$$\frac{\partial g_1}{\partial r} = 1, \quad \frac{\partial g_2}{\partial r} = 0$$

$$\Rightarrow \quad m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta = \lambda_1 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = mgr\sin\theta, \quad \frac{\partial g_1}{\partial \theta} = 0, \quad \frac{\partial g_2}{\partial \theta} = -(R+a)$$

$$\Rightarrow \quad mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mgr\sin\theta = -\lambda_2(R+a) \qquad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = I\dot{\phi}, \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = I\ddot{\phi}, \quad \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\frac{\partial g_1}{\partial \phi} = 0, \quad \frac{\partial g_2}{\partial \phi} = a$$

$$\Rightarrow \quad I\ddot{\phi} = \lambda_2 a \qquad (3)$$

From $g_1(r,\theta,\phi)$ we have $r = R+a$

$$\dot{r} = \ddot{r} = 0$$

$$(1) \Rightarrow -m(R+a)\dot{\theta}^2 + mg\cos\theta = \lambda_1 \qquad (4)$$

$$(2) \Rightarrow \quad m(R+a)^2\ddot{\theta} - mg(R+a)\sin\theta = -\lambda_2(R+a)$$

$$\Rightarrow m(R+a)\ddot{\theta} - mg\sin\theta = -\lambda_2 \qquad (5)$$

From $g_2(r,\theta,\phi) = a(\phi-\theta) - R\theta = 0$

$$\Rightarrow \quad \phi = \frac{(R+a)\theta}{a} \Rightarrow \ddot{\phi} = \left(\frac{R+a}{a}\right)\ddot{\theta}$$

substitute in (3) $\Rightarrow \quad \ddot{\theta} = \dfrac{\lambda_2 a^2}{I(R+a)}$

substitute into (5) to get

$$m(R+a)\frac{\lambda_2 a^2}{I(R+a)} - mg\sin\theta = -\lambda_2$$

$$\lambda_2\left(1+\frac{ma^2}{I}\right) = mg\sin\theta$$

$$\Rightarrow \boxed{\lambda_2 = \frac{mg\sin\theta}{\left(1+\frac{ma^2}{I}\right)}}$$

For a rolling disk $I = \frac{1}{2}ma^2 \Rightarrow \lambda_2 = \frac{1}{3}mg\sin\theta$

Substitute above result for λ_2 into

$$\ddot{\theta} = \frac{\lambda_2 a^2}{I(R+a)} = \frac{mg\sin\theta}{\left(1+\frac{ma^2}{I}\right)} \cdot \frac{a^2}{I(R+a)} = \frac{g\sin\theta}{\left(1+\frac{I}{ma^2}\right)}\frac{1}{(R+a)}$$

as in previous example, $\ddot{\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$, so

$$\int \dot{\theta}\, d\dot{\theta} = \frac{g}{\left(1+\frac{I}{ma^2}\right)(R+a)} \int \sin\theta\, d\theta$$

$$\dot{\theta}^2 = \frac{2g}{\left(1+\frac{I}{ma^2}\right)(R+a)}(1-\cos\theta)$$

where we assumed that $\dot{\theta}=0$ when $\theta=0$

substituting this into (4) gives

$$-m(R+a)\dot{\theta}^2 + mg\cos\theta = \lambda_1$$

$$\lambda_1 = \frac{-m(R+a)\,2g(1-\cos\theta)}{\left(1+\frac{I}{ma^2}\right)(R+a)} + mg\cos\theta$$

$$\boxed{\lambda_1 = \frac{-2mg(1-\cos\theta)}{\left(1+\frac{I}{ma^2}\right)} + mg\cos\theta}$$

for a disk with $I = \frac{1}{2}ma^2$

$$\lambda_1 = -\frac{4}{3}mg(1-\cos\theta) + mg\cos\theta$$

$$\lambda_1 = \frac{mg}{3}(7\cos\theta - 4)$$

Generalized forces:

in r direction, gives normal force

$$N = \lambda_1 \frac{\partial g_1}{\partial r} + \lambda_2 \frac{\partial g_2}{\partial r} = \lambda_1$$

we see that N becomes unphysically negative when $\cos\theta = \frac{4}{7}$, or when $\theta = 55.15°$

(this is a larger angle than in previous example without rolling)

this would cause disk to leave surface when $\theta \geq 55.15°$.

But another problem occurs before this happens

in ϕ direction, the generalized force is the frictional torque

$$\tau_\phi = F_f a = \lambda_1 \frac{\partial g_1}{\partial \phi} + \lambda_2 \frac{g_2}{\partial \phi} = \lambda_2 a$$

$$F_f a = \frac{1}{3} a m g \sin\theta$$

$$\Rightarrow \quad F_f = \frac{1}{3} m g \sin\theta$$

in θ direction we get the same frictional torque

$$\tau_\theta = \lambda_1 \frac{\partial g_1}{\partial \theta} + \lambda_2 \frac{g_2}{\partial \theta} = -\lambda_2 (R+a)$$

work done by τ_θ as disk moves $\delta\theta$ should equal word done by τ_ϕ as disk moves $\delta\phi$

$$\tau_\theta d\theta = \lambda_2 (R+a)\,\delta\theta = \tau_\phi \,\delta\phi = \lambda_2 a \,\delta\phi$$

$$\Rightarrow \quad (R+a)\,\delta\theta = a\,\delta\phi$$

but this is indeed true by constraint g_2

Now, small disk rolls without slipping, the frictional force F_f is due to static friction.

~~We also have~~ $\cancel{\underbrace{F_f^{max}}_{\text{maximum possible friction}} = \mu_s N}$ ~~μ_s = coefficient of static friction~~

~~or~~ $$\cancel{\mu_s = \frac{F_f^{max}}{N} = \frac{\lambda_2}{\lambda_1} = \frac{\frac{1}{3} m g \sin\theta_{max}}{\frac{mg}{3}(7\cos\theta_{max} - 4)}}$$

If we define the ratio

$$\mu \equiv \frac{F_f}{N}$$

then we know that $\mu \le \mu_s$ the coefficient of static friction, $F_f^{max} = \mu_s N$

Now

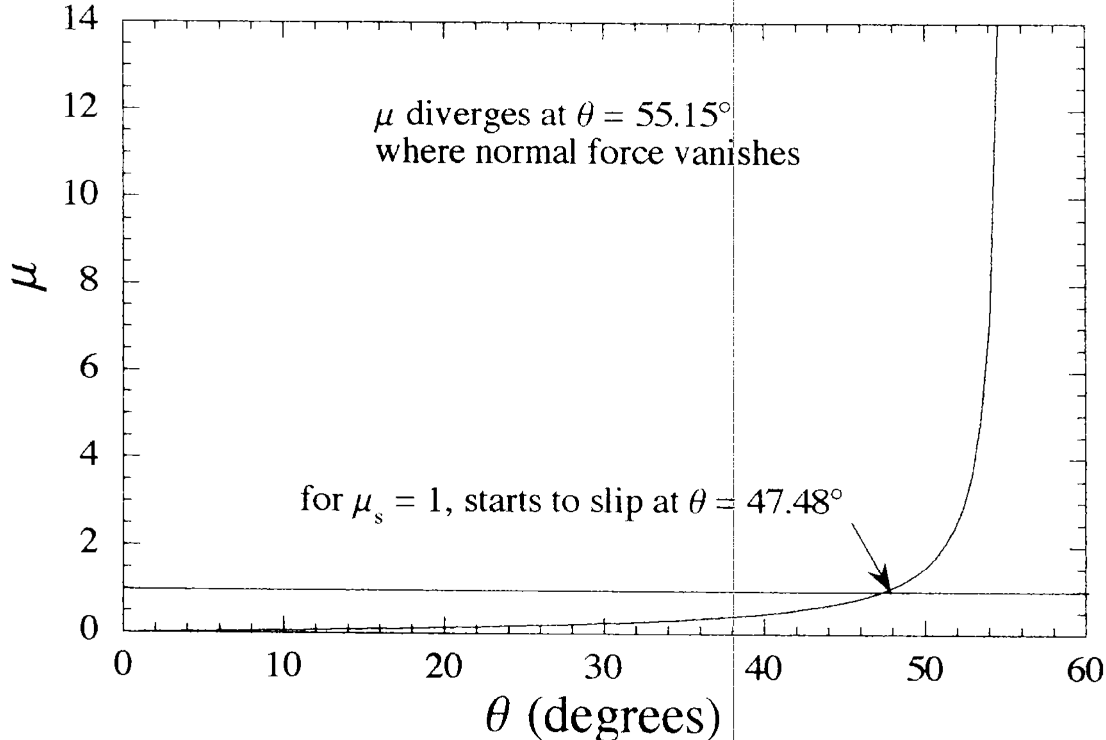
$$\mu = \frac{F_f}{N} = \frac{\lambda_2}{\lambda_1} = \frac{\frac{1}{3} mg \sin\theta}{\frac{mg}{3}(7\cos\theta - 4)}$$

$$\mu = \frac{\sin\theta}{7\cos\theta - 4}$$

As θ increases, μ increases until it eventually reaches the value μ_s. For larger values of θ, the frictional force can no longer be large enough to maintain the no slipping constraint. The disk will start to slip as it falls down the hemispherical surface.

Note: This onset of slipping always occurs <u>before</u> the vanishing of the normal force, which causes the disk to leave the surface.

this is because $\mu_s = \frac{F_f}{N}$ must always occur before $N \to 0$, assuming μ_s is finite.



14
12
10
8
6
4
2
0
0
10
20
30
40
50
60
μ
θ (degrees)
μ diverges at $\theta = 55.15°$
where normal force vanishes
for $\mu_s = 1$, starts to slip at $\theta = 47.48°$