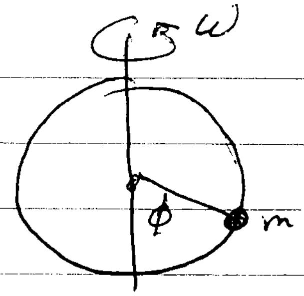
prob 7-21



$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \left(R^2 \dot{\phi}^2 + R^2 \sin^2\phi \, \omega^2 \right)$$

$$U = -mgR\cos\phi$$

$$\mathcal{L} = T - U = \frac{1}{2} m R^2 \left(\dot{\phi}^2 + \omega^2 \sin^2\phi \right) + mgR\cos\phi$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mR^2 \dot{\phi} \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = mR^2 \ddot{\phi}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \omega^2 m R^2 \sin\phi \cos\phi - mgR\sin\phi$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = mR^2 \ddot{\phi} - \omega^2 m R^2 \sin\phi\cos\phi + mgR\sin\phi = 0$$

$$\boxed{\ddot{\phi} - \omega^2 \sin\phi\cos\phi + \frac{g}{R}\sin\phi = 0}$$

stationary solutions ϕ_0 when $\ddot{\phi} = 0$

$$\Rightarrow \phi_0 = 0 \text{ or } \pi$$

$$\text{or} \quad \cos\phi_0 = \frac{g}{\omega^2 R}$$

To analyze the stability of each of the three stationary solutions, we linearize in small deviations $\phi = \phi_0 + \delta\phi$

$$\sin(\phi_0 + \delta\phi) = \sin\phi_0 \cos\delta\phi + \cos\phi_0 \sin\delta\phi$$
$$\simeq \sin\phi_0 + \cos\phi_0 \, \delta\phi$$

$$\cos(\phi_0 + \delta\phi) = \cos\phi_0 \cos\delta\phi - \sin\phi_0 \sin\delta\phi$$
$$\simeq \cos\phi_0 - \sin\phi_0 \, \delta\phi$$

for ① $\quad \phi_0 = 0$, $\quad \sin(\phi_0+\delta\phi) \simeq \cos\phi_0\,\delta\phi = \delta\phi$

$\cos(\phi_0+\delta\phi) \simeq \cos\phi_0 = 1$

$$\ddot{\delta\phi} - \omega^2\,\delta\phi + \frac{g}{R}\,\delta\phi = 0$$

$$\ddot{\delta\phi} + \left(\frac{g}{R} - \omega^2\right)\delta\phi = 0$$

If $\omega^2 < g/R$ then have harmonic oscillation with angular freq $\omega_0 = \sqrt{\frac{g}{R} - \omega^2}$

for ② $\quad \phi_0 = \pi$, $\quad \sin(\phi_0+\delta\phi) \approx -\delta\phi$

$\cos(\phi_0+\delta\phi) \approx -1$

$$\ddot{\delta\phi} - \left(\omega^2 + \frac{g}{R}\right)\delta\phi = 0 \quad \Rightarrow \quad \delta\phi \sim e^{\lambda t}$$

$$\lambda = \sqrt{\omega^2 + \frac{g}{R}}$$

small deviations grow $\Rightarrow$ state at $\phi_0 = \pi$ is unstable

for ③ $\quad \cos\phi_0 = \frac{g}{\omega^2 R}$ $\quad \sin(\phi_0+\delta\phi) = \sin\phi_0 + \frac{g}{\omega^2 R}\,\delta\phi$

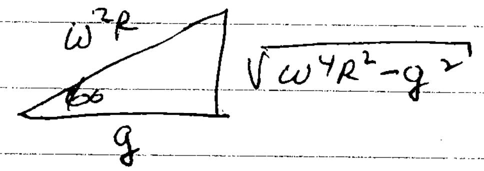
$\cos(\phi_0+\delta\phi) = \frac{g}{\omega^2 R} - \sin\phi_0\,\delta\phi$

$$\ddot{\delta\phi} - \omega^2\left(\sin\phi_0 + \frac{g}{\omega^2 R}\,\delta\phi\right)\left(\frac{g}{\omega^2 R} - \sin\phi_0\,\delta\phi\right) + \frac{g}{R}\left(\sin\phi_0 + \frac{g}{\omega^2 R}\,\delta\phi\right) = 0$$

dropping terms $O(\delta\phi^2)$

$$\ddot{\delta\phi} - \omega^2\sin\phi_0\,\frac{g}{\omega^2 R} + \omega^2\sin\phi_0\,\delta\phi - \omega^2\left(\frac{g}{\omega^2 R}\right)^2\delta\phi + \frac{g}{R}\sin\phi_0 + \frac{g^2}{\omega^2 R^2}\,\delta\phi = 0$$

$$\Rightarrow \quad \delta\ddot{\phi} + \omega^2 \sin^2\phi_0 \, \delta\phi = 0$$



$$\sin^2\phi_0 = \frac{\omega^4R^2 - g^2}{\omega^4R^2} = 1 - \frac{g^2}{\omega^4R^2}$$

$$\omega^2 \sin^2\phi_0 = \omega^2 - \frac{g^2}{\omega^2R^2}$$

$$\Rightarrow \quad \delta\ddot{\phi} + \left(\omega^2 - \frac{g^2}{\omega^2R^2}\right)\delta\phi = 0$$

if $\omega^2 > g^2/\omega^2R^2$ or $\omega^2 > g/R$

$\Rightarrow$ harmonic oscillations with

angular freq $\omega_0 = \sqrt{\omega^2 - g^2/\omega^2R^2}$

Conclusion: Define $\omega_c \equiv \sqrt{g/R}$

If $\omega < \omega_c$ then stable state is $\phi_0 = 0$
and harmonic motion has angular freq

$$\omega_0 = \sqrt{\omega_c^2 - \omega^2} = \omega\sqrt{\left(\frac{\omega_c}{\omega}\right)^2 - 1}$$

If $\omega > \omega_c$ then stable state is $\cos\phi_0 = \dfrac{g}{\omega^2R}$
and harmonic motion has angular freq

$$\omega_0 = \sqrt{\omega^2 - \frac{\omega_c^4}{\omega^2}} = \omega\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^4}$$

Velocity dependent forces

Newtonian method use forces + vector equations - must include all forces of constraint in eqn of motion

Lagrangian method uses kinetic + potential energy, and generalized coordinates - all scalar quantities. Forces of constraint either eliminated by choice of coordinates, or by Lagrange multipliers

Lagrangian method - usually easier to derive equations of motion. - But can only be used when forces arise from either ~~of~~ a potential energy or a holonomic constraint.

Newtons method - good for any type of force

If we have forces that depend on velocity, for example ~~drag~~ viscous drag forces, these in general cannot be written as either from potential energy or holonomic constraint - so Lagrange's method would not work.

~~There is one important exception. If the velocity dependent force F_i^{NC} can be written in terms of a potential $U^{NC}(q, \dot{q}, t)$ such that~~

$$\cancel{F_i^{NC} = \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_i} - \frac{\partial U}{\partial q_i}}$$

~~(Note this is a generalization of what we mean by a potential energy)~~

There is one important exception, If the non-conservative velocity dependent force can be written in the following form

$$F_i^{NC} = \frac{d}{dt}\left(\frac{\partial U^{NC}}{\partial \dot{q}_i}\right) - \frac{\partial U^{NC}}{\partial q_i}$$

where U^{NC} is the generalized potential for the velocity dependent force, then from Lagrange's equ with non conservative force we get

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = F_i^{NC} = \frac{d}{dt}\left(\frac{\partial U^{NC}}{\partial \dot{q}_i}\right) - \frac{\partial U^{NC}}{\partial q_i}$$

or $$\frac{d}{dt}\left(\frac{\partial (\mathcal{L} - U^{NC})}{\partial \dot{q}_i}\right) - \frac{\partial (\mathcal{L} - U^{NC})}{\partial q_i} = 0$$

So F^{NC} can be incorporated into $\mathcal{L}$ by just including ~~it as~~ U^{NC} as part of the potential energy U.

$$\mathcal{L} \Rightarrow T - U \rightarrow \mathcal{L} = T - U - U^{NC}$$

and then $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$ as before

A very important example of this is the Lorentz force

$$\vec{F} = e(\vec{E} + \vec{v} \times \vec{B})$$

which one can show comes from the generalized potential

$$U = eV - e\vec{v} \cdot \vec{A}$$

where $\left\{ \begin{array}{l} \vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} = \vec{\nabla} \times \vec{A} \end{array} \right.$

scalar + vector potentials V, $\vec{A}$

$$\frac{\partial U}{\partial \dot{x}_i} = -eA_i \qquad \frac{d}{dt}\frac{\partial U}{\partial \dot{x}_i} = -e\frac{dA_i}{dt} = -e\left(\sum_j \frac{\partial A_i}{\partial x_j}\dot{x}_j + \frac{\partial A_i}{\partial t}\right)$$

$$\frac{\partial U}{\partial x_i} = e\frac{\partial V}{\partial x_i} - e\vec{v}\cdot\frac{\partial \vec{A}}{\partial x_i} \qquad = -e\left(\vec{v}\cdot\vec{\nabla}A_i + \frac{\partial A_i}{\partial t}\right)$$

so $$\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}_i}\right) - \frac{\partial U}{\partial x_i} = -e\frac{\partial A_i}{\partial t} - e\,\vec{v}\cdot\vec{\nabla}A_i - e\frac{\partial V}{\partial x_i} + e\vec{v}\cdot\frac{\partial \vec{A}}{\partial x_i}$$

$$= e\left(-\frac{\partial V}{\partial x_i} - \frac{\partial A_i}{\partial t}\right) + e\left(\vec{v}\cdot\frac{\partial \vec{A}}{\partial x_i} - \vec{v}\cdot\vec{\nabla}A_i\right)$$

first term is just eE_i where $\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$

compare second term to $e(\vec{v}\times\vec{B})_i = e\left(\vec{v}\times(\vec{\nabla}\times\vec{A})\right)_i$

$$= e\sum_{jklm}\varepsilon_{ijk}v_j\,\varepsilon_{klm}\nabla_l A_m$$

$$= e\sum_{jklm}(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,v_j\nabla_l A_m$$

$$= e\left(\vec{v}\cdot\nabla_i\vec{A} - \vec{v}\cdot\vec{\nabla}A_i\right)$$

$$= e\left(\vec{v}\cdot\frac{\partial \vec{A}}{\partial x_i} - \vec{v}\cdot\nabla\vec{A}_i\right)$$ same as 2^{nd} term.

So $$e\left(\vec{E} + \vec{v}\times\vec{B}\right) = \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}_i}\right) - \frac{\partial U}{\partial x_i}$$

when $U = eV - e\vec{v}\cdot\vec{A}$