# Conservation Laws + Generalized Momentum

For cartesian coordinates we had

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = p_i \qquad i^{th} \text{ component of momentum } p_i = m\dot{x}_i$$

↑ assumes $U$ is independent of $\dot{x}_i$

For a generalized coordinate $q_i$, we can define the generalized momentum by

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = p_i \quad \leftarrow \text{momentum "conjugate" to } q_i$$

Lagrange's eqn's can then be written as

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \quad \Rightarrow \quad \frac{dp_i}{dt} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

If $\mathcal{L}$ does not depend on the variable $q_i$, ie $\mathcal{L}$ is invarient to shifts in the coordinate $q_i$, then $\frac{\partial \mathcal{L}}{\partial q_i} = 0$ and so

$$\frac{dp_i}{dt} = 0 \Rightarrow p_i = \text{constant}$$

then the generalized momentum corresponding to $q_i$ is conserved.

If $q_i = x_i$ the usual cartesian coordinate, then the above is just the law of conservation of linear momentum, ie if no external force in $i^{th}$ direction, $F_i = 0 \Rightarrow \frac{\partial U}{\partial x_i} = 0 \Rightarrow \mathcal{L}$ indep of $x_i$ $\Rightarrow p_i =$ constant.

If $q_i$ is an angular variable, the above implies the conservation of angular momentum. Proof: For an infintesmal rotation $\delta\vec{\theta}$, the change in $\vec{r}$ is given by $\delta\vec{r} = \delta\vec{\theta} \times \vec{r}$.

If $\mathcal{L}$ is invariant with respect to rotations about the axis given by $\delta\vec{\theta}$, then, viewing $\mathcal{L}$ as a function of $\vec{r}$ and $\dot{\vec{r}}$ we can write

Change in $\mathcal{L}$ for a small $\delta\vec{\theta}$ is

$$\delta\mathcal{L} = \sum_i \left( \frac{\partial \mathcal{L}}{\partial x_i}\delta x_i + \frac{\partial \mathcal{L}}{\partial \dot{x}_i}\delta \dot{x}_i \right) = 0$$ since $\mathcal{L}$ invariant to $\delta\vec{\theta}$ rotation

Now $\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = p_i$ linear momentum

From Lagrange's equ $\frac{dp_i}{\partial t} - \frac{\partial L}{\partial x_i} = 0$ we get

$$\frac{\partial L}{\partial x_i} = +\dot{p}_i$$

So

$$\delta\mathcal{L} = \sum_i \left( \dot{p}_i\,\delta x_i + p_i\,\delta\dot{x}_i \right) = 0$$

$$\delta\mathcal{L} = \dot{\vec{p}}\cdot\delta\vec{r} + \vec{p}\cdot\delta\dot{\vec{r}}$$

substitute in $\delta\vec{r} = \delta\vec{\theta}\times\vec{r}$

$$\delta\mathcal{L} = \dot{\vec{p}}\cdot(\delta\vec{\theta}\times\vec{r}) + p\cdot(\delta\vec{\theta}\times\dot{\vec{r}})$$

use $\bar{A}\cdot(\bar{B}\times\bar{C}) = \vec{B}\cdot(\bar{C}\times\vec{A}) = \bar{C}\cdot(\vec{A}\times\vec{B})$
to rewrite as

$$\delta\mathcal{L} = \delta\vec{\theta}\cdot(r\times\dot{\vec{p}}) + \delta\vec{\theta}\cdot(\dot{\vec{r}}\times\vec{p})$$

$$= \delta\vec{\theta}\cdot\frac{d}{dt}(\vec{r}\times\vec{p}) = 0$$

$$= \delta\vec{\theta}\cdot\frac{d\vec{L}}{dt} = 0$$ where $\vec{L} = \vec{r}\times\vec{p}$ is angular momentum

$$\frac{d}{dt}(\delta\vec{\theta}\cdot\vec{L}) = 0$$

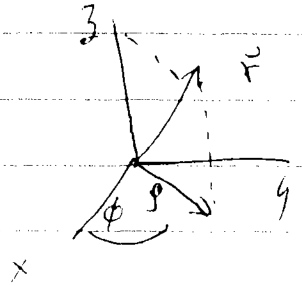
$\Rightarrow$ component of $\vec{L}$ in direction $\delta\vec{\theta}$ is constant.

Conservation of component of $\vec{L}$ in direction of $\delta\vec{\theta}$

If $\mathcal{L}$ is invarient to rotations along any axis, then

$\vec{L} =$ const - all components of $\vec{L}$ are conserved

Another way to see this. If, for exaple, $\mathcal{L}$ is invarient for rotations about $\hat{z}$, then, in cylindrical coordinates $\mathcal{L}$ is independent of the variable $\phi$.




$$\Rightarrow P_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \text{const}$$

But $\mathcal{L}$ depends on $\dot{\phi}$ through the kinetic energy

$$T = \frac{1}{2} m (\dot{z}^2 + \dot{\rho}^2 + \rho^2 \dot{\phi}^2) \quad \text{in cylindrical coordinates}$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial T}{\partial \dot{\phi}} = m \rho^2 \dot{\phi} = m\rho (\rho \dot{\phi}) = m \rho v_\phi = L_z$$

since
$$L_z = \hat{z} \cdot (\vec{r} \times \vec{p}) = m \hat{z} \cdot (\vec{r} \times \vec{v})$$
$$= m \vec{v} \cdot (\hat{z} \times \vec{r}) = m \vec{v} \cdot (\hat{z} \times (z\hat{z} + \rho\hat{\rho}))$$
$$= m \vec{v} \cdot (\rho \hat{z} \times \hat{\rho}) = m \vec{v} \cdot (\rho \hat{\phi})$$
$$L_z = m \rho v_\phi$$

$$\Rightarrow p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = L_z \quad \text{is conserved}$$

conjugate momentum to $\phi$ is just $p_\phi = L_z$ angular momentum about $\hat{z}$.

## Kinetic Energy + Conservation of Energy

kinetic energy in generalized coordinates.

in terms of cartesian coordinates $x_i$

$$T = \frac{1}{2} \sum_{\alpha=1}^{n} \sum_{i=1}^{3} m_\alpha \dot{x}_{\alpha,i}^2$$

$n$ particles $\alpha = 1, \ldots, n$
of mass $m_\alpha$,
position $\vec{r}_\alpha$

If these $x_{\alpha i}$ can be expressed in terms of $s$ generalized coordinates $q_j$, $j = 1, \ldots, s$

$$\Rightarrow \quad x_{\alpha i} = x_{\alpha i}(q_j, t)$$

and generalized velocities

$$\dot{x}_{\alpha i} = \sum_{j=1}^{s} \frac{\partial x_{\alpha i}}{\partial q_j} \dot{q}_j + \frac{\partial x_{\alpha, i}}{\partial t}$$

Substitute into $T$ to get

$$T = \frac{1}{2} \sum_{\alpha=1}^{n} \sum_{i j k=1}^{3} m_\alpha \left( \frac{\partial x_{\alpha i}}{\partial q_j} \dot{q}_j + \frac{\partial x_{\alpha i}}{\partial t} \right) \left( \frac{\partial x_{\alpha k}}{\partial q_k} \dot{q}_k + \frac{\partial x_{\alpha i}}{\partial t} \right)$$

$$= \sum_{jk} a_{jk} \dot{q}_j \dot{q}_k + \sum_j b_j \dot{q}_j + c$$

where $a_{jk} = \sum_i \sum_\alpha \frac{1}{2} m_\alpha \left( \frac{\partial x_{\alpha i}}{\partial q_j} \right) \left( \frac{\partial x_{\alpha k}}{\partial q_k} \right)$

$$b_j = \sum_k \sum_i \sum_\alpha m_\alpha \left( \frac{\partial x_{\alpha i}}{\partial q_j} \right) \left( \frac{\partial x_{\alpha i}}{\partial t} \right)$$

$$c = \sum \sum m_\alpha \left( \frac{\partial x_{\alpha i}}{} \right) \left( \frac{\partial x_{\alpha i}}{} \right)$$

Now if time $t$ does not explicitly appear in the coordinate transformations, ie

$$\frac{\partial x_{\alpha i}}{\partial t} = 0$$

then

$$\boxed{T = \sum_{jk} a_{jk} \dot{q}_j \dot{q}_k}$$

Another result concerning kinetic energy can be derived as follows:

$$\frac{\partial T}{\partial \dot{q}_\ell} = \sum_k a_{\ell k} \dot{q}_k + \sum_j a_{j\ell} \dot{q}_j$$

$$\sum_\ell \dot{q}_\ell \frac{\partial T}{\partial \dot{q}_\ell} = \sum_{\ell k} a_{\ell k} \dot{q}_k \dot{q}_\ell + \sum_{j\ell} a_{j\ell} \dot{q}_j \dot{q}_\ell = 2T$$

so

$$\boxed{2T = \sum_\ell \dot{q}_\ell \left(\frac{\partial T}{\partial \dot{q}_\ell}\right)}$$

We use these two results to discuss energy conservation as follows:

~~Consider the total time derivative of $\mathcal{L}[q_i, \dot{q}_i, t]$~~

$$\cancel{\frac{d\mathcal{L}}{dt} = \sum_i \left[\frac{\partial \mathcal{L}}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt}\right] + \frac{\partial \mathcal{L}}{\partial t}}$$

Consider the total time derivative of $\mathcal{L}[q_i, \dot{q}_i, t]$

$$\frac{d\mathcal{L}}{dt} = \sum_i \left( \frac{\partial \mathcal{L}}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} \right) + \frac{\partial \mathcal{L}}{\partial t}$$

If all forces are conservative, then Lagrange's equations give

$$\frac{d}{dt}\left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

substitute into above to get

$$\frac{d\mathcal{L}}{dt} = \sum_i \left( \left( \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) \dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \ddot{q}_i \right) + \frac{\partial \mathcal{L}}{\partial t}$$

$$= \sum_i \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i \right) + \frac{\partial \mathcal{L}}{\partial t}$$

$$\Rightarrow \quad \frac{d}{dt} \left( \mathcal{L} - \sum_i \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) = \frac{\partial \mathcal{L}}{\partial t}$$

or using our definition of canonical momentum $p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$

$$\frac{d}{dt} \left( \mathcal{L} - \sum_i \dot{q}_i p_i \right) = \frac{\partial \mathcal{L}}{\partial t}$$

Define the Hamiltonian $\mathcal{H} \equiv \sum_i \dot{q}_i p_i - \mathcal{L}$

the above becomes

$$-\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{L}}{\partial t}$$

For the case where $\mathcal{L}$ does not explicitly depend on time ie the potential energy $U$ does not vary with time,

$$\frac{\partial \mathcal{L}}{\partial t} = 0$$

In this case we have $\frac{d\mathcal{H}}{dt} = 0$

ie $\mathcal{H}$ = constant

the Hamiltonian is a conserved quantity or a "constant of motion".

Now if ~~the potential U does not depend explicitly on velocities $\dot{q}_i$ and~~ the coordinate ~~than~~ transformation between generalized and rectangular coordinates $x_{\alpha i}(q_j, t)$ do not explicitly depend on $t$,

ie $x_{\alpha i} = x_{\alpha i}(q_j)$

Then we have our earlier results concerning the kinetic energy

$$T = \sum_{jk} a_{jk}\, \dot{q}_j\, \dot{q}_k \quad \text{and so} \quad 2T = \sum_{\ell} \dot{q}_\ell \frac{\partial T}{\partial \dot{q}_\ell}$$

~~Hamiltons equations of motion~~

To summarize, we see that physical symmetries are associated with a corresponding symmetry of the Lagrangian, which are associated with a conserved quantity

spatial invarience $\Rightarrow$ $\mathcal{L}$ indep of $x_i$ $\Rightarrow$ linear momentum conserved

rotational invarience $\Rightarrow$ $\mathcal{L}$ indep of angle $\phi$ $\Rightarrow$ angular momentum conserved

temporal invarience $\Rightarrow$ $\mathcal{L}$ indep of $t$ $\Rightarrow$ energy conserved

## Hamilton's Equations of Motion

we defined the canonical momentum by

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

since Lagrange's eqn of motion are satisfied

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

we then have

$$\dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i}$$

We can imagine solving the equations

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i}[q_j, \dot{q}_j, t]$$

to write $\dot{q}_i$ in terms of the $q_j, p_j, t$
and so to write the Hamiltonian

$$\mathcal{H} = \sum_i \dot{q}_i p_i - \mathcal{L}$$

as a function of $q_i, p_i, t$

The Hamiltonian $\mathcal{H}[q_i, p_i, t]$ is regarded as a function of the generalized coordinates and momenta $q_i, p_i$
The Lagrangian $\mathcal{L}[q_i, \dot{q}_i, t]$ is regarded as a function of the generalized coordinates and velocities $q_i, \dot{q}_i$

Now consider the change in $\mathcal{H}$, as one varies $q_i, p_i, t$

$$d\mathcal{H} = d\left[\sum_i \dot{q}_i p_i - \mathcal{L}[q_i, \dot{q}_i, t]\right]$$

$$= \sum_i \left[\dot{q}_i\, dp_i + p_i\, d\dot{q}_i - \frac{\partial \mathcal{L}}{\partial q_i} dq_i - \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i\right] - \frac{\partial \mathcal{L}}{\partial t} dt$$

use $\frac{\partial \mathcal{L}}{\partial q_i} = \dot{p}_i$ and $\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = p_i$ to get

$$d\mathcal{H} = \sum_i \left[\dot{q}_i\, dp_i + \cancel{p_i\, d\dot{q}_i} - \dot{p}_i\, dq_i - \cancel{p_i\, d\dot{q}_i}\right] - \frac{\partial \mathcal{L}}{\partial t} dt$$

and if the potential $\mathcal{U}$ does not depend explicitly on the velocities $\dot{q}_i$ or time $t$, then

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial (T-\mathcal{U})}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i}$$

and we get

$$\mathcal{H} = \sum_i \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L} = \sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} - \mathcal{L}$$

$$= 2T - \mathcal{L} = 2T - (T - \mathcal{U})$$

$$\mathcal{H} = T + \mathcal{U} = \text{total mechanical energy}$$

We thus conclude that when

① The coordinate transformations $x_{\alpha i}(q_i)$ do not explicitly depend on time $t$

and when

② The potential energy $\mathcal{U}$ does not depend explicitly on the generalized velocities $\dot{q}_i$ or time $t$,

then the Hamiltonian

$$\mathcal{H} = \sum_i \dot{q}_i p_i - \mathcal{L}$$

is equal to the total mechanical energy $T + \mathcal{U}$ and it is a constant of the motion.

$$d\mathcal{H} = \sum_i \left[ \dot{q}_i \, dp_i - \dot{p}_i \, dq_i \right] - \frac{\partial \mathcal{L}}{\partial t} dt$$

compare with

$$d\mathcal{H} = \sum_i \left[ \frac{\partial \mathcal{H}}{\partial q_i} dq_i + \frac{\partial \mathcal{H}}{\partial p_i} dp_i \right] + \frac{\partial \mathcal{H}}{\partial t} dt \qquad (*)$$

we conclude

$$\boxed{\frac{dq_i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i} \quad \text{and} \quad \frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q_i}} \quad \text{and} \quad \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$$

these are Hamilton's equations of motion.

moreover we can write $(*)$, dividing by $dt$, as

$$\frac{d\mathcal{H}}{dt} = \sum_i \left[ \frac{\partial \mathcal{H}}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial \mathcal{H}}{\partial p_i} \frac{dp_i}{dt} \right] + \frac{\partial \mathcal{H}}{\partial t}$$

substituting in Hamilton's eqn of motion then gives

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t}$$

$\Rightarrow$ if Hamiltonian is <u>not</u> an explicit function of time, ie $\partial \mathcal{H}/\partial t = 0$, then $\mathcal{H}$ is a constant of the motion, ie $d\mathcal{H}/dt = 0$.

[and if $U$ is independent of $\dot{q}_i$, and $x_{\alpha_j}(q_i)$ is indep of $t$, then $\mathcal{H} = E$ total mechanical energy]

If there are $n$ generalized coordinates $q_i$ $i = 1, \dots, n$ then Hamiltons equations

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$$

are a set of $2n$ first order differential equations for the $2n$ functions $q_i$ and $p_i$

Compare this to Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

which are a set of $n$ second order differential equations for the $n$ functions $q_i$

Since $n$ 2nd order diff eqs are mathematically equivalent to $2n$ 1st order diff eqs, what is the advantage of Hamiltonian over Lagrangian mechanics? In fact, it is often easier to write the eqs of motion using Lagrange's rather than Hamiltons Equations.
One answer lies in looking at the situation where $\mathcal{L}$ is independent of a particular coordinate $q_k$.
In this case, although $\partial \mathcal{L}/\partial q_k = 0$, one still has $n$ 2nd order differential equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

to solve for the $n$ $q_i(t)$. Except for the simplification that for the $q_k$ equation we can easily do one integration

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \text{const}$$

Even though $\mathcal{L}$ is indep of $q_i$, it still depends on $\dot{q}_i$
we have not really reduced the number of degrees of freedom.

In the Hamiltonian formulation however, we get

$$\dot{p}_k = \frac{\partial \mathcal{L}}{\partial q_k} = 0$$

$$\Rightarrow p_k \equiv \alpha_k \text{ a constant}$$ , the momentum $p_k$ is conserved.

If $\mathcal{L}$ is indep of $q_i$, then so is $\mathcal{H}$. Since $p_k \equiv \alpha_k$ is constant, ~~we have that the~~ the only degrees of freedom that $\mathcal{H}$ depends on are $n-1$ other $q_i$ and the $n-1$ other $p_i$

$\mathcal{H}$ is $\mathcal{H}[q_1, q_2, \ldots, q_{k-1}, q_{k+1}, \ldots q_n, p_1, \ldots, p_{k-1}, \alpha_k, p_{k+1}, \ldots p_n$

We have therefore eliminated the two degrees of freedom $q_k$ and $p_k$ from the problem. The remaining ~~2n-2~~ Hamilton eqns are $2n-2$ ~~equations for the remaining~~ 1st order diff eqns for the remaining $2n-2$ variables $q_i, p_i$ $i \neq k$.

Thus, whenever $\mathcal{L}$ is indep of a coordinate $q_k$
$\Rightarrow$ exist corresponding constant of the motion $p_k$,
$\Rightarrow$ can reduce degrees of freedom in the Hamiltonian formulation.

One can next try to find transformations among the variables $q_i, p_i$ ie $q_i'(q_j, p_j)$ and $p_i'(q_j, p_j)$ such that when expressed in terms of $q_i'$ and $p_i'$, $H$ is independent of some particular $q_k'$. If one could find a particular transformation such that $H$ was independent of ~~all~~ all the $q_k'$, then one would in effect have solved the problem by identifying $n$ constants of the motion. If one can do this, then the dynamics is solved because the ~~coordin~~ momenta $p_i' \equiv \alpha_i$ are all constants, and $H(q_i', p_i')$ depends only on these constants. Then

$$\dot{q}_k = \frac{\partial H}{\partial p_k} = \frac{\partial H}{\partial \alpha_k} \quad \text{can be simply integrated}$$

to solve for all the $q_k(t)$.

The method of solving dynamical problems by trying to find such transformations is known as Hamilton-Jacobi theory.

Another advantage of the Hamiltonian formulation is that it is readily extended to other fields of physics. Quantum mechanics was based on the Hamiltonian formulation. Similarly statistical mechanics, in which one does averages over the degrees of freedom $q_i, p_i$ (which define "phase space") is based on the Hamiltonian formulation.