Examples

① particle in 1-D in potential $U(x)$

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - U(x)$$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$ ordinary linear momentum

$$\Rightarrow \dot{x} = p/m$$

$$\mathcal{H} = \dot{x} p - \mathcal{L}[x, \dot{x}]$$

$$\mathcal{H} = \left(\frac{p}{m}\right) p - \frac{1}{2} m \left(\frac{p}{m}\right)^2 + U(x)$$

$$= \frac{p^2}{m} - \frac{1}{2}\frac{p^2}{m} + U(x)$$

$$\boxed{\mathcal{H} = \frac{1}{2}\frac{p^2}{m} + U(x)}$$ = total mechanical energy $T + U$

$$\dot{p} = \frac{-\partial H}{\partial x} = -\frac{\partial U}{\partial x} \qquad \left.\begin{array}{l} \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \end{array}\right\} \Rightarrow m\ddot{x} = -\frac{\partial U}{\partial x}$$

② Galilean transformation of a free particle

$$T = \frac{1}{2} m \dot{x}^2 \qquad U = 0$$

Define new coordinate $x' = x - ut$
x' measures coordinate in a frame that moves with velocity u with respect to the x coord frame.

$x' = x - ut \Rightarrow x = x' + ut$

$\dot{x} = \dot{x}' + u$

$T = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m (\dot{x}' + u)^2$

$\mathcal{L} = T - U = \frac{1}{2} m (\dot{x}' + u)^2$

canonical momentum $p' = \frac{\partial \mathcal{L}}{\partial \dot{x}'} = m(\dot{x}' + u) = m\dot{x} = p$

so momentum in the new coordinate system is same as in old

$\mathcal{H}' = \dot{x}' p' - \mathcal{L}[x', \dot{x}']$

$p' = m(\dot{x}' + u)$

$\Rightarrow \frac{p'}{m} - u = \dot{x}'$

$= \left(\frac{p'}{m} - u\right) p' - \frac{1}{2} m \left(\frac{p'}{m} - u + u\right)^2$

$= \frac{p'^2}{m} - up - \frac{1}{2} \frac{p'^2}{m}$

$$\boxed{\mathcal{H}' = \frac{1}{2} \frac{p'^2}{m} - up}$$

Note H' is not the total mechanical energy

since $T = \frac{1}{2} \frac{p^2}{m} = \frac{1}{2} \frac{p'^2}{m}$

$H' = E - up$

Since coord transf depended explicitly on t, there is no reason H' is total energy.

H' is still conserved — (this we know is true because E and p are conserved)

(3) Canonical momentum of a charged particle

For a charge e in an electromagnetic field we had the generalized potential

$$U = eV - e\vec{v}\cdot\vec{A} \qquad \vec{v} \equiv \dot{\vec{r}}$$

$$\mathcal{L} = T - U = \frac{1}{2}m\sum_{i=1}^{3}\dot{x}_i^2 - eV + e\sum_{i=1}^{3}\dot{x}_i A_i$$

Canonical momentum

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = m\dot{x}_i + eA_i \quad , \text{ or } \vec{p} = m\vec{v} + e\vec{A}$$

$$\Rightarrow \quad \dot{x}_i = \frac{p_i - eA_i}{m}$$

$$\mathcal{H} = \sum_i \dot{x}_i p_i - \mathcal{L}$$

$$= \sum_i \left(\frac{p_i - eA_i}{m}\right) p_i - \frac{1}{2}m\sum_i \left(\frac{p_i - eA_i}{m}\right)^2 + eV$$

$$- e\sum_i \left(\frac{p_i - eA_i}{m}\right) A_i$$

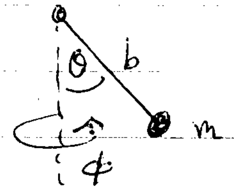
$$= \frac{p^2 - e\vec{p}\cdot\vec{A}}{m} - \frac{1}{2}\left(\frac{p^2 - 2e\vec{p}\cdot\vec{A} + e^2A^2}{m}\right) + eV$$

$$- \left(\frac{e\vec{p}\cdot\vec{A} - e^2A^2}{m}\right)$$

$$= \frac{1}{2}\frac{p^2}{m} - \frac{e\vec{p}\cdot\vec{A}}{m} + \frac{1}{2}\frac{e^2A^2}{m} + eV$$

$$\boxed{\mathcal{H} = \frac{1}{2}\frac{(\vec{p} - e\vec{A})^2}{m} + eV}$$

(4) Spherical pendulum



$T = \frac{1}{2} m v^2 = \frac{1}{2} m (b^2 \dot{\theta}^2 + b^2 \sin^2\theta \, \dot{\phi}^2)$

$U = -mgb\cos\theta$

$$\mathcal{L} = \frac{1}{2} m (b^2 \dot{\theta}^2 + b^2 \sin^2\theta \, \dot{\phi}^2) + mgb\cos\theta$$

canonical momentum

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m b^2 \dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{m b^2}$$

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m b^2 \sin^2\theta \, \dot{\phi} \Rightarrow \dot{\phi} = \frac{p_\phi}{m b^2 \sin^2\theta}$$

$$\mathcal{H} = \dot{\theta} p_\theta + \dot{\phi} p_\phi - \mathcal{L}$$

$$= \frac{p_\theta^2}{m b^2} + \frac{p_\phi^2}{m b^2 \sin^2\theta} - \frac{1}{2} m b^2 \left(\frac{p_\theta^2}{m^2 b^4} + \sin^2\theta \frac{p_\phi^2}{m^2 b^4 \sin^4\theta} \right)$$

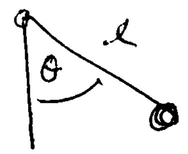
$$- mgb\cos\theta$$

$$\mathcal{H} = \frac{p_\theta^2}{2mb^2} + \frac{p_\phi^2}{2mb^2\sin^2\theta} - mgb\cos\theta$$

equ of motion:

$$\begin{cases} \dot{\theta} = \frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{p_\theta}{mb^2} \qquad \dot{\phi} = \frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{mb^2\sin^2\theta} \\ \dot{p}_\theta = -\frac{\partial \mathcal{H}}{\partial \theta} = \frac{p_\phi^2 \cos\theta}{mb^2 \sin^3\theta} - mgb\sin\theta \\ \dot{p}_\phi = -\frac{\partial \mathcal{H}}{\partial \phi} = 0 \end{cases}$$

7 - 24



$\ell = \alpha t$

note: $\mathcal{L}$ depends explicitly on time $\Rightarrow$ ~~H~~ not conserved

$$\mathcal{L} = \frac{1}{2} m(\ell^2\dot{\theta}^2 + \dot{\ell}^2) + mg\ell\cos\theta$$

$$= \frac{1}{2} m(\alpha^2 t^2 \dot{\theta}^2 + \alpha^2) + mg\ell\cos\theta$$

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m\alpha^2 t^2 \dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{m\alpha^2 t^2}$$

$$\mathcal{H} = \frac{p_\theta^2}{m\alpha^2 t^2} - \frac{1}{2} m\left(\alpha^2 t^2 \frac{p_\theta^2}{m^2\alpha^4 t^4} + \alpha^2\right) - mg\ell\cos\theta$$

$$\boxed{\mathcal{H} = \frac{p_\theta^2}{2m\ell^2} - \frac{1}{2} m\alpha^2 - mg\ell\cos\theta}$$

Compare this to $E = T + U$

$$= \frac{1}{2} m(\alpha^2 t^2 \dot{\theta}^2 + \alpha^2) - mg\ell\cos\theta$$

$$= \frac{1}{2}\frac{p_\theta^2}{m\ell^2} + \frac{1}{2} m\alpha^2 - mg\ell\cos\theta$$

$$\neq \mathcal{H}$$

coordinate transf between x, y and θ are not indep of time

$$\begin{cases} x = \ell\sin\theta = \alpha t\sin\theta \\ y = -\ell\cos\theta = -\alpha t\cos\theta \end{cases}$$

7-32) $U = -\frac{k}{r}$ $\quad v =$

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2 \right)$$

$$\mathcal{L} = T - U = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2 \right) + \frac{k}{r}$$

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \qquad \dot{r} = p_r/m$$

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m r^2 \dot{\theta} \qquad \dot{\theta} = p_\theta / m r^2$$

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m r^2 \sin^2\theta \, \dot{\phi} \qquad \dot{\phi} = p_\phi / m r^2 \sin^2\theta$$

$$\mathcal{H} = \frac{p_r^2}{m} + \frac{p_\theta^2}{m r^2} + \frac{p_\phi^2}{m r^2 \sin^2\theta} - \frac{1}{2} m \left(\frac{p_r^2}{m^2} + r^2 \frac{p_\theta^2}{m^2 r^4} + r^2 \sin^2\theta \frac{p_\phi^2}{m^2 r^4 \sin^4\theta} \right) - \frac{k}{r}$$

$$\mathcal{H} = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2 m r^2} + \frac{p_\phi^2}{2 m r^2 \sin^2\theta} - \frac{k}{r}$$

$$\dot{p}_r = -\frac{\partial \mathcal{H}}{\partial r} = \frac{p_\theta^2}{m r^3} + \frac{p_\phi^2}{m r^3 \sin^2\theta} - \frac{k}{r^2}$$

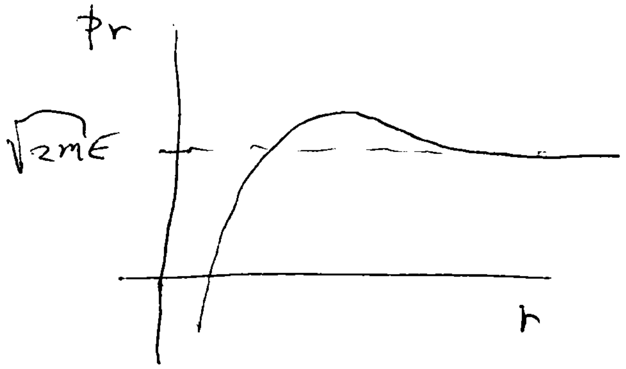
$$\left\{ \begin{array}{l} \dot{p}_\theta = \dfrac{-\partial \mathcal{H}}{\partial \theta} = \dfrac{p_\phi^2 \cos\theta}{m r^2 \sin^3\theta} \\ \dot{p}_\phi = 0 \\ \dot{r} = \dfrac{\partial \mathcal{H}}{\partial p_r} = \dfrac{p_r}{m}, \quad \dot{\theta} = \dfrac{\partial \mathcal{H}}{\partial p_\theta} = \dfrac{p_\theta}{m r^2}, \quad \dot{\phi} = \dfrac{\partial \mathcal{H}}{\partial p_\phi} = \dfrac{p_\phi}{m r^2 \sin^2\theta} \end{array} \right.$$

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{c}{2mr^2\sin^2\theta} - \frac{k}{r} = E$$

variables r, θ, p_r, p_θ

constraint $E=0$ fixes one of the varibles in terms of the other 3. for exapn

$$p_r = \sqrt{2mE - \frac{p_\theta^2}{r^2} - \frac{c}{r^2\sin^2\theta} + \frac{2mk}{r}}$$



fixed θ, p_θ

Poisson bracket

7-30) $[g,h] = \sum_k \left(\frac{\partial g}{\partial q_k} \frac{\partial h}{\partial p_k} - \frac{\partial g}{\partial p_k} \frac{\partial h}{\partial q_k} \right)$

$$\frac{dg}{dt} = \sum_k \frac{\partial g}{\partial q_k} \dot{q}_k + \frac{\partial g}{\partial t}$$

$$= \sum_k \frac{\partial g}{\partial q_k} \frac{\partial H}{\partial p_k} + \frac{\partial g}{\partial t}$$

$g(q_k, p_k, t)$

$$\Rightarrow \frac{dg}{dt} = \sum_k \left(\frac{\partial g}{\partial q_k} \dot{q}_k + \frac{\partial g}{\partial p_k} \dot{p}_k \right) + \frac{\partial g}{\partial t}$$

$$= \sum_k \left(\frac{\partial g}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial g}{\partial p_k} \frac{\partial H}{\partial q_k} \right) + \frac{\partial g}{\partial t}$$

$$= [g, H] + \frac{\partial g}{\partial t}$$

If $g = q \Rightarrow \dot{q} = [q, H] + \frac{\partial q}{\partial t}$ but $\frac{\partial q}{\partial t} = 0$

$$\dot{q} = [q, H]$$

$g = p \Rightarrow \dot{p} = [p, H]$ as $\frac{\partial p}{\partial t} = 0$

$$[p_i, p_j] = \sum_k \left(\frac{\partial p_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) = 0 \text{ as } \frac{\partial p_i}{\partial q_k} = 0$$

similarly $[q_i, q_j] = 0$

$$[q_i, p_j] = \sum_k \left(\frac{\partial q_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right)$$

$$= \sum_k (\delta_{ik} \delta_{jk} - 0) = \delta_{ij}$$

If g does not explicitly depend on time, ie $\frac{\partial g}{\partial t} = 0$, and $[g, H] = 0$ then

$$\frac{dg}{dt} = [g, H] + \frac{\partial g}{\partial t} = 0$$

$$\Rightarrow \frac{dg}{dt} = 0 \Rightarrow g = \text{constant}$$

Poisson brackets act just like commutators in Q.M.