Two Body Motion for a central force

two particles m_1 at $\vec{r}_1$, m_2 at $\vec{r}_2$

force of m_1 on m_2 depends only on the distance $|\vec{r}_1 - \vec{r}_2|$ and points in direction from m_1 to m_2. Such a force is called a "central force"

$$\vec{F} = f(r)\,\hat{r} \quad \text{where } \vec{r} = \vec{r}_1 - \vec{r}_2$$

$$= -\frac{\partial U}{\partial r}\cdot\hat{r} \qquad U \text{ is potential for } \vec{F}$$

Instead of using coordinates $\vec{r}_1$ and $\vec{r}_2$, it is convenient to use instead the

$$\left.\begin{array}{ll} \text{relative separation} & \vec{r} = \vec{r}_1 - \vec{r}_2 \\ \text{center of mass} & \vec{R} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \end{array}\right\}$$

We can invert the above equations to solve for $\vec{r}_1$ and $\vec{r}_2$ in terms of $\vec{R}$ and $\vec{r}$

$$\left.\begin{array}{rl} \vec{r} &= \vec{r}_1 - \vec{r}_2 \\ \dfrac{(m_1+m_2)\vec{R}}{m_2} &= \dfrac{m_1}{m_2}\vec{r}_1 + r_2 \end{array}\right\} \text{add} \Rightarrow \vec{r}_1\left(1 + \frac{m_1}{m_2}\right) = \vec{r} + \frac{m_1+m_2}{m_2}\vec{R}$$

$$\Rightarrow \vec{r}_1 = \left(\frac{m_2}{m_1+m_2}\right)\left[\vec{r} + \frac{m_1+m_2}{m_2}\vec{R}\right]$$

$$\vec{r}_1 = \vec{R} + \left(\frac{m_2}{m_1+m_2}\right)\vec{r}$$

$$\vec{r}_2 = \vec{r}_1 - \vec{r} = \vec{R} + \left(\frac{m_2}{m_1+m_2}\right)\vec{r} - \vec{r}$$

$$= \vec{R} + \frac{m_2 - (m_1+m_2)}{m_1+m_2}\vec{r}$$

$$= \vec{R} - \left(\frac{m_1}{m_1+m_2}\right)\vec{r}$$

So
$$\begin{cases} \vec{r}_1 = \vec{R} + \left(\frac{m_2}{m_1+m_2}\right)\vec{r} \\ \vec{r}_2 = R - \left(\frac{m_1}{m_1+m_2}\right)\vec{r} \end{cases}$$

The Lagrangian for the system is

$$\mathcal{L} = \frac{1}{2}m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2 \dot{\vec{r}}_2^{\,2} - U(r)$$

substitute for $\vec{r}_1$ and $\vec{r}_2$ in terms of $\vec{r}$ and $\vec{R}$ to get

$$\mathcal{L} = \frac{1}{2}m_1\left[\dot{\vec{R}} + \frac{m_2}{M}\dot{\vec{r}}\right]^2 + \frac{1}{2}m_2\left[\dot{\vec{R}} - \frac{m_1}{M}\dot{\vec{r}}\right]^2 - U(r$$

where $M \equiv m_1 + m_2$

$$\mathcal{L} = \frac{1}{2}m_1\left[\dot{\vec{R}}^2 + \frac{m_2^2}{M^2}\dot{\vec{r}}^{\,2} + 2\frac{m_2}{M}\dot{\vec{R}}\cdot\dot{\vec{r}}\right]$$

$$+ \frac{1}{2}m_2\left[\dot{\vec{R}}^2 + \frac{m_1^2}{M^2}\dot{\vec{r}}^{\,2} - 2\frac{m_1}{M}\dot{\vec{R}}\cdot\dot{\vec{r}}\right] - U$$

$$= \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \frac{1}{2}\frac{m_1 m_2}{M^2}(m_1+m_2)\dot{\vec{r}}^{\,2} - U(r)$$

$$= \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)\dot{\vec{r}}^{\,2} - U(r)$$

Define $\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$ or $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$

$$\boxed{\mathcal{L} = \tfrac{1}{2} M \dot{\vec{R}}^2 + \tfrac{1}{2} \mu \dot{\vec{r}}^2 - U(r)}$$

Consider Lagrange's eqn of motion for $\vec{R}$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{R}_i}\right) - \frac{\partial \mathcal{L}}{\partial R_i} = 0 \Rightarrow M\ddot{R}_i = 0 \quad \text{or} \quad M\dot{R}_i = \text{constant}$$

$$\Rightarrow M\dot{\vec{R}} = M\left(\frac{m_1 \dot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2}{m_1 + m_2}\right) = m_1 \dot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2 = \vec{p}_1 + \vec{p}_2$$

$$= \vec{P}_{total} = \text{constant}$$

So total momentum is conserved.

Lagranges eqn of motion for $\vec{r}$ are

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}_i}\right) - \frac{\partial \mathcal{L}}{\partial r_i} = 0 \Rightarrow \mu \ddot{r}_i + \frac{\partial U}{\partial r}\frac{\partial r}{\partial r_i} = 0$$

these equations do <u>not</u> couple at all to $\vec{R}$.

So center of mass motion $\vec{R}(t)$ is decoupled from relative separation motion $\vec{r}(t)$.

Hence forth we ignore center of mass motion and focus only on $\vec{r}(t)$.

$\mathcal{L} = \frac{1}{2}\mu\dot{\vec{r}}^{\,2} - U(r)$

motion of $\vec{r}(t)$ is the same as that of a point particle of mass $\mu = \frac{m_1 m_2}{m_1+m_2}$ in a central force field pointing radially to the origin given by the potential energy $U(r)$.

μ is called the "reduced mass"

Once we solve for $\vec{r}(t)$, we know at once the motion of the two particles $\vec{r}_1(t) = \vec{R} + \frac{m_2}{M}\vec{r}$
and $\vec{r}_2(t) = \vec{R} - \frac{m_1}{M}\vec{r}$

Equation of Motion

Because $\mathcal{L}$ has rotational symmetry (since U depends only on $|\vec{r}| = r$ is spherically symmetric) we know angular momentum $\vec{L}$ is conserved

$\Rightarrow \vec{L} = \vec{r}\times\vec{p} = \text{const}$ where $\vec{p} = \mu\dot{\vec{r}}$

Consider $\vec{L}\cdot\vec{r} = (\vec{r}\times\vec{p})\cdot\vec{r} = 0$
(since $\vec{r}\times\vec{p}$ is orthogonal to both $\vec{r}$ and to $\vec{p}$)

$\Rightarrow$ ~~random~~ trajectory of $\vec{r}(t)$ has no component in the direction of $\vec{L}$, ie $\vec{r}(t)$ lies entirely in the plane orthogonal to the constant $\vec{L}$.

$\Rightarrow$ We can parameterize the trajectory $\vec{r}(t)$ in terms of polar coordinates r, θ, in the plane of the motion.

In polar coordinates $\dot{\vec{r}} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$

$$\boxed{\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)}$$

$\mathcal{L}$ is independent of θ $\Rightarrow$ conjugate momentum conserv

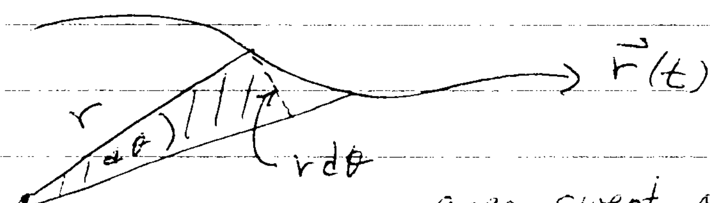
$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0$

$\Rightarrow$ $p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \boxed{\mu r^2 \dot{\theta} \equiv \ell}$ a constant

= angular momentum

called "first integral of the motion"

<u>Keplers 2nd law</u>

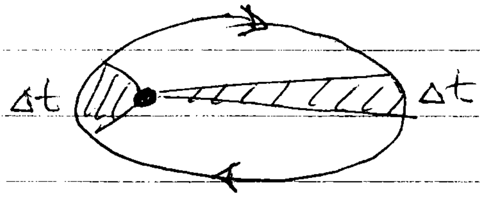


area swept out is

$dA = \frac{1}{2} r(r\,d\theta) = \frac{1}{2} r^2 d\theta$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2}\frac{\ell}{\mu} = \text{const}$$

rate of area swept out as particle moves is constant.



$\Rightarrow$ particle moves faster in its orbit when it is closest to the origin.

Similarly, totaly energy $E = T + U$ is conserved.

$$E = \tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + U(r) \quad = \text{constant}$$

But since we found $\mu r^2 \dot{\theta} = \ell$ is constant,

$$\dot{\theta} = \frac{\ell}{\mu r^2}$$

$$\Rightarrow \quad E = \tfrac{1}{2}\mu\dot{r}^2 + \tfrac{1}{2}\mu r^2 \frac{\ell^2}{\mu^2 r^4} + U(r)$$

$$\boxed{E = \tfrac{1}{2}\mu\dot{r}^2 + \tfrac{1}{2}\frac{\ell^2}{\mu r^2} + U(r)} = \text{constant}$$

We could have derived this from the Lagrange equation for r

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = 0 \Rightarrow \mu\ddot{r} - \mu r\dot{\theta}^2 + \frac{\partial U}{\partial r} = 0$$

$$\mu r^2\dot{\theta} = \ell \Rightarrow \boxed{\mu\ddot{r} - \frac{\ell^2}{\mu r^3} + \frac{\partial U}{\partial r} = 0}$$

use trick we saw before:

$$\ddot{r} = \frac{d\dot{r}}{dr}\frac{dr}{dt} = \frac{d\dot{r}}{dr}\dot{r}$$

$$\Rightarrow \int \mu \dot{r}\, d\dot{r} = \int \left(\frac{\ell^2}{\mu r^3} - \frac{\partial U}{\partial r}\right) dr$$

$$\Rightarrow \quad \tfrac{1}{2}\mu\dot{r}^2 = -\frac{1}{2}\frac{\ell^2}{\mu r^2} - U + E$$

E is const of integration

$$\Rightarrow \quad \tfrac{1}{2}\mu\dot{r}^2 + \tfrac{1}{2}\ell^2/\mu r^2 + U = E$$

motion of $r(t)$ looks just like motion of a particle in one dimension in ~~a potential~~ an effective potential

$$U_{eff}(r) = U(r) + \frac{1}{2}\frac{\ell^2}{\mu r^2}$$

So a finite angular momentum ℓ looks like an effective force $F_{eff} = -\frac{\partial}{\partial r}\left(\frac{1}{2}\frac{\ell^2}{\mu r^2}\right) = \frac{\ell^2}{\mu r^3}$ on the radial coord.

This $F_{eff} = \frac{\ell^2}{\mu r^3} = \frac{(\mu r^2 \dot{\theta})^2}{\mu r^3} = \mu r \dot{\theta}^2$

is just the fictitious "centrifugal force" which arises from the fact that the coordinate r is like an ordinary Cartesian coordinate, only measured in a non-inertial rotating frame of reference.

Solution to Eqn of motion

Using $E = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r)$

we can write

$$\dot{r} = \frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}\left(E - U - \frac{\ell^2}{2\mu r^2}\right)}$$

$\Rightarrow$

$$\boxed{t = \pm\int \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - U - \frac{\ell^2}{2\mu r^2}\right)}}}$$

since the integrand is a known function of r, we can always in principle do the integration, and then invert the result to get $r(t)$, given a fixed E and ℓ.

One then integrates $\dot{\theta} = \dfrac{\ell}{\mu r^2}$

$$\theta(t) = \int dt \, \frac{\ell}{\mu [r(t)]^2}$$

to find $\theta(t)$.

One thus has in principle solved for the trajectory $(r(t), \theta(t))$ for a given conserved energy E and angular momentum ℓ.

One can similarly find an integral solution to the trajectory as parameterized by θ, ie $r(\theta)$

$$\dot{r} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\dot{\theta} = \frac{dr}{d\theta}\frac{\ell}{\mu r^2} = \pm\sqrt{\frac{2}{\mu}\left(E - U - \frac{\ell^2}{2\mu r^2}\right)}$$

$$\Rightarrow \quad d\theta = \frac{\pm \, dr \, \ell}{\mu r^2 \sqrt{\frac{2}{\mu}\left(E - U - \frac{\ell^2}{2\mu^2}\right)}}$$

$$\boxed{\theta = \pm \int \frac{dr \, \ell}{r^2 \sqrt{2\mu\left(E - U - \frac{\ell^2}{2\mu^2}\right)}}}$$

can always integrate and then invert to get $r(\theta)$.