General properties of motion

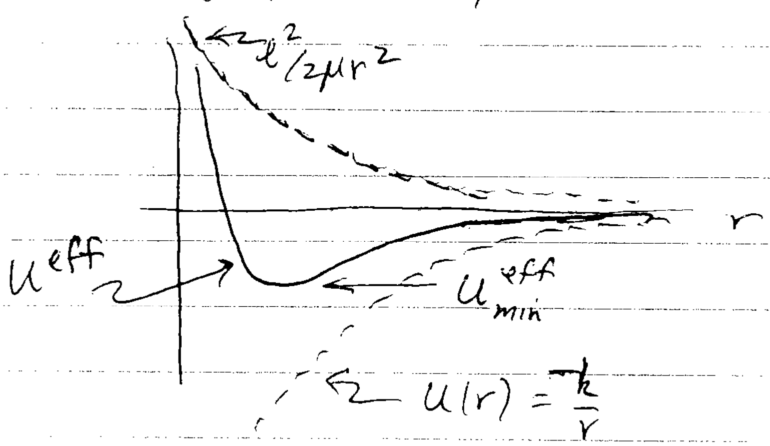
We had that the total mechanical energy was conserved

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r)$$

we can view this just like motion in one dimension - given by the coordinate r - in an effective potential

$$U^{eff}(r) = U(r) + \frac{1}{2}\frac{\ell^2}{\mu r^2}$$

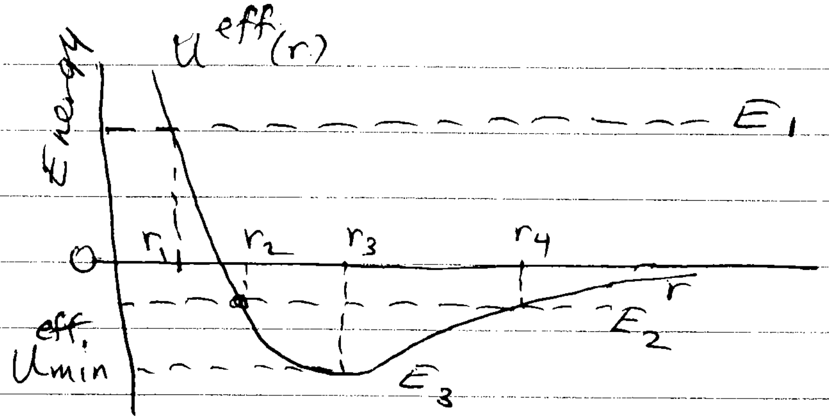
For the gravitational force, $U(r) = -\frac{k}{r}$, and we can graph the pieces as follows



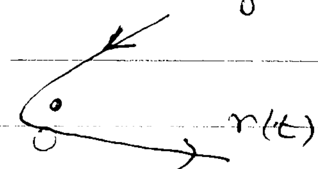
We see that $U^{eff}(r)$ has a minimum value at some particular value of r. ~~This is general for other choices of $U(r)$ that are attractive~~

For U^{eff} to have min, $U(r)$ must be attractive

We can therefore use conservation of energy to classify the possible motions as follows:

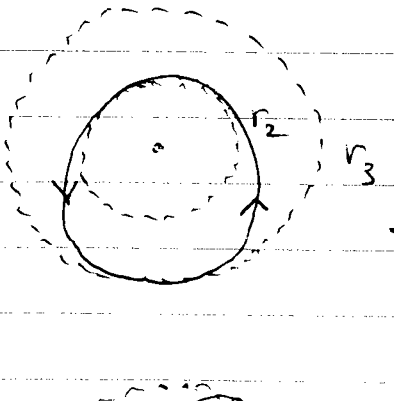


i) For an energy $E_1 > 0$, there is a single turning point at r_1. This represents a particle coming in from $r = \infty$, scattering off the origin, and going back out to $r = \infty$

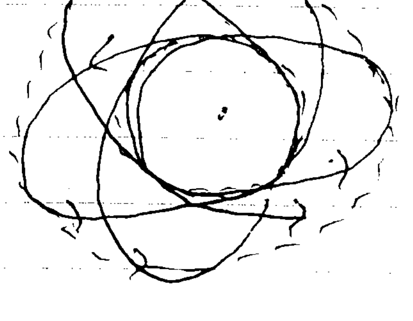


This is an "unbound" state

ii) For an energy $U^{eff}_{min} < E_2 < 0$, there are two turning points r_2 and r_3. The motion is confined between the radii r_2 and r_3.



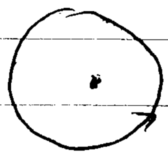
this motion may be either as a closed orbit



or as an open orbit

these are "bound" states

iii) If $E_3 = U^{eff}_{min}$ then the system must stay at the potential minimum r_3, ie the radius of motion $r(t)$ is constant in time. This means a circular orbit



r_3

iv) $E < U^{eff}_{min}$ would require a negative kinetic energy, and so is not allowed classically.

For the Gravitational potential $U = -\frac{k}{r}$,

$$U^{eff} = -\frac{k}{r} + \frac{1}{2}\frac{\ell^2}{\mu r^2}$$

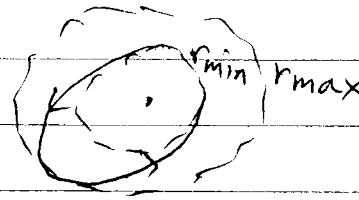
has its minimum at

$$\frac{\partial U^{eff}}{\partial r} = \frac{k}{r^2} - \frac{2}{2}\frac{\ell^2}{\mu r^3} = 0$$

$$\Rightarrow \quad \mu k r = \ell^2 \quad \text{or} \quad r = \frac{\ell^2}{\mu k}$$

$$U^{eff}_{min} = -\frac{k\,\mu k}{\ell^2} + \frac{1}{2}\frac{\ell^2}{\mu}\frac{\mu^2 k^2}{\ell^4} = -\frac{\mu k^2}{2\ell^2}$$

For the bound ~~state~~ of case (ii) $U_{min}^{eff} < E < 0$, we can determine whether the orbit is closed or open as follows:



we found earlier that

$$d\theta = \frac{\pm\, dr\, \ell}{\mu r^2 \sqrt{\frac{2}{\mu}\left(E - U - \frac{\ell^2}{2\mu^2}\right)}}$$

First note that since $\ell = \mu r^2 \dot{\theta}$ is constant, then $\dot{\theta}$ can never change signs, therefore $\theta(t)$ must always decrease or increase monotonically with time

now integrate the above going from r_{min} to r_{max} (the smallest and largest radial distances of the orbit) and then back from r_{max} to r_{min}. The angular change in this is

$$\Delta\theta = 2\int_{r_{min}}^{r_{max}} \frac{\ell\, dr}{r^2\sqrt{2\mu\left(E - U - \frac{\ell^2}{2\mu r^2}\right)}}$$

(since change in θ from r_{min} to r_{max} is same as from r_{max} to r_{min})

If this $\Delta\theta$ is a rational fraction of 2π, ie

$$\Delta\theta = 2\pi\,\frac{a}{b} \quad \text{for integer } a \text{ and } b$$

Then the orbit will close – after b periods of motion from r_{min} to r_{max} and back to r_{min}, the particle will have made a revolutions about the origin.

<u>Planetary Motion</u> $\quad U(r) = \frac{-k}{r}$

We now focus on the problem where $U(r)$ is the gravitational potential $-k/r$.
We had

$$\theta(r) = \int \frac{l\,dr}{r^2\sqrt{2\mu\left(E - U - \frac{l^2}{2\mu r^2}\right)}} + \text{const}$$

$$\theta(r) = \int \frac{l\,dr}{r^2\sqrt{2\mu\left(E + \frac{k}{r} - \frac{l^2}{2\mu r^2}\right)}} + \text{const}$$

change variable of integration to $u = l/r$

$$du = -\frac{l}{r^2}\,dr$$

$$\theta = -\int du \frac{1}{\sqrt{2\mu\left(E + \frac{k}{l}u - \frac{u^2}{2\mu}\right)}} + \text{const}$$

~~define origin of θ so that $\theta = 0$ corresponds to r_{min}.~~

$$\theta = -\int \frac{du}{\sqrt{2\mu E + \frac{2\mu k u}{\ell} - u^2}} + \text{const}$$

change integration variables to $v = u - \frac{\mu k}{\ell}$ $\qquad dv = du$

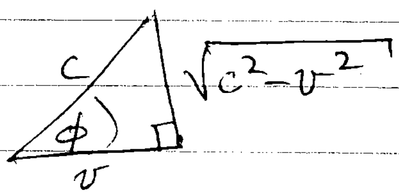
$$v^2 = u^2 - \frac{2\mu k u}{\ell} + \frac{\mu^2 k^2}{\ell^2}$$

$$\theta = -\int \frac{dv}{\sqrt{2\mu E + \frac{\mu^2 k^2}{\ell^2} - v^2}} + \text{const}$$

call $\quad c^2 \equiv 2\mu E + \mu^2 k^2/\ell^2$

$$\theta = -\int \frac{dv}{\sqrt{c^2 - v^2}} + \text{const}$$

can do this integral by trigonometric substitution



$$\sin\phi = \frac{\sqrt{c^2 - v^2}}{c}$$

$$\Rightarrow \cos\phi \, d\phi = -\frac{v}{c} \frac{dv}{\sqrt{c^2 - v^2}} = -\cos\phi \frac{dv}{\sqrt{c^2 - v^2}}$$

$$\Rightarrow d\phi = \frac{-dv}{\sqrt{c^2 - v^2}}$$

$\theta = \int d\phi + \text{const}$ $\qquad$ call the const $= \theta_0$

$$\Rightarrow \theta - \theta_0 = \phi$$

$$\cos(\theta - \theta_0) = \cos\phi = \frac{v}{c} = \frac{u - \mu k/\ell}{\sqrt{2\mu E + \mu^2 k^2/\ell^2}}$$

$$\cos(\theta-\theta_0) = \frac{u - \mu k/\ell}{\frac{\mu k}{\ell}\sqrt{1+\frac{2E\ell^2}{\mu k^2}}} = \frac{\frac{\ell u}{\mu k} - 1}{\sqrt{1+\frac{2E\ell^2}{\mu k^2}}}$$

$$\cos(\theta-\theta_0) = \frac{\left(\frac{\ell^2}{\mu k r} - 1\right)}{\sqrt{1+\frac{2E\ell^2}{\mu k^2}}}$$

Note, when $\theta = \theta_0$, then

$$\cos(0) = 1 = \frac{\frac{\ell^2}{\mu k r} - 1}{\sqrt{1+\frac{2E\ell^2}{\mu k^2}}}$$

$$\Rightarrow \quad 1 + \frac{2E\ell^2}{\mu k^2} = \left(\frac{\ell^2}{\mu k r} - 1\right)^2 = \frac{\ell^4}{\mu^2 k^2 r^2} + 1 - \frac{2\ell^2}{\mu k r}$$

$$\Rightarrow \quad E = \frac{\ell^2}{2\mu r^2} - \frac{k}{r}$$ $\leftarrow$ this is just expression for energy when kinetic energy vanishes, ie when $\dot{r} = 0$.

So $\theta = \theta_0$ corresponds to r being one of the two turning points. As θ increases above θ_0, $\cos(\theta-\theta_0)$ decreases, $\Rightarrow$ r increases. Hence $\theta = \theta_0$ corresponds to the turning point at the smallest radial distance r_{min}.

So solution is

$$\cos(\theta - \theta_0) = \frac{\left(\frac{\ell^2}{\mu k r} - 1\right)}{\sqrt{1 + \frac{2E\ell^2}{\mu k^2}}}$$

where $\theta = \theta_0$ gives $r = r_{min}$.

Henceforth we choose $\theta_0 = 0$, so that $\theta = 0$ is point of closest approach to the origin, $r = r_{min}$

If we define $\alpha \equiv \frac{\ell^2}{\mu k}$ and $\varepsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}$

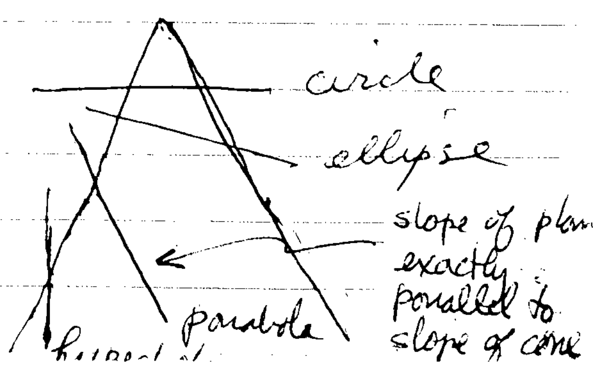
above solution is

$$\cos\theta = \frac{\frac{\alpha}{r} - 1}{\varepsilon}$$

$$\Rightarrow \boxed{\frac{\alpha}{r} = 1 + \varepsilon\cos\theta}$$

This is equation of conic section with one focus at the origin.

conic sections are curves formed by intersection of a plane with a cone



$$\frac{\alpha}{r} = 1 + \varepsilon \cos\theta \qquad \varepsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}$$

For $\varepsilon = 0$ ie $E = \frac{-\mu k^2}{2\ell^2} = U^{eff}_{min}$

we have a circular orbit

For $0 < \varepsilon < 1$ ie $U^{eff}_{min} < E < 0$

we have elliptical orbit, ε is the eccentricity

For $\varepsilon = 1$, ie $E = 0$

we have a parabolic orbit

For $\varepsilon > 1$, ie $E > 0$

we have a hyperbolic orbit