when the coordinate axes are assumed to be given one sometimes writes the vector $\vec{r} = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3$ as the triple of numbers (x_1, x_2, x_3) ← row vector

or $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ column vector

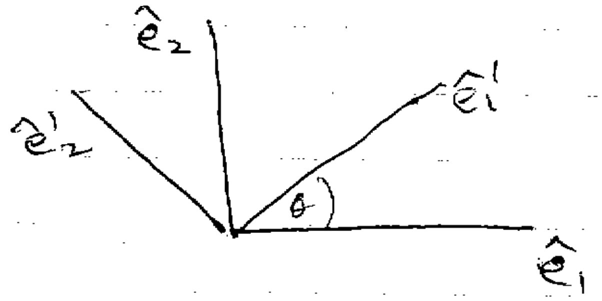
this notation will be convenient when dealing with matrices.

Rotation Matrix

Suppose we know the coordinates (x_1, x_2, x_3) of a vector with respect to a set of basis vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$, ie $\vec{r} = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3$

What will be the coordinates of $\vec{r}$ (x_1', x_2', x_3') with respect to a different set of basis vectors $\hat{e}_1', \hat{e}_2', \hat{e}_3'$ that are rotated with respect to the first?

Given (x_1, x_2, x_3), how do we find (x_1', x_2', x_3')



← example in 2D.
$\hat{e}_1', \hat{e}_2'$ are rotated by angle θ with respect to $\hat{e}_1, \hat{e}_2$

Consider x_1' - this is the projection of $\vec{r}$ onto unit vector $\hat{e}_1'$, so

$$x_1' = \hat{e}_1' \cdot \vec{r} = \hat{e}_1' \cdot (x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3)$$
$$= (\hat{e}_1' \cdot \hat{e}_1) x_1 + (\hat{e}_1' \cdot \hat{e}_2) x_2 + (\hat{e}_1' \cdot \hat{e}_3) x_3$$

Similarly,
$$x_2' = (\hat{e}_2' \cdot \hat{e}_1) x_1 + (\hat{e}_2' \cdot \hat{e}_2) x_2 + (\hat{e}_2' \cdot \hat{e}_3) x_3$$
$$x_3' = (\hat{e}_3' \cdot \hat{e}_1) x_1 + (\hat{e}_3' \cdot \hat{e}_2) x_2 + (\hat{e}_3' \cdot \hat{e}_3) x$$

or, with summation notation

$$x_i' = \sum_j (\hat{e}_i' \cdot \hat{e}_j) x_j$$ (sum over j is implie

call $\boxed{\lambda_{ij} \equiv (\hat{e}_i' \cdot \hat{e}_j) = \cos(\hat{e}_i', \hat{e}_j)}$

cosine angle between $\hat{e}_i'$ ad $\hat{e}_j$

$\equiv$ "direction cosine" of $\hat{e}_i'$ with respect to $\hat{e}_j$

In matrix notation

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

matrix multiplicati

$$x_i' = \sum_j \lambda_{ij} x_j$$

or

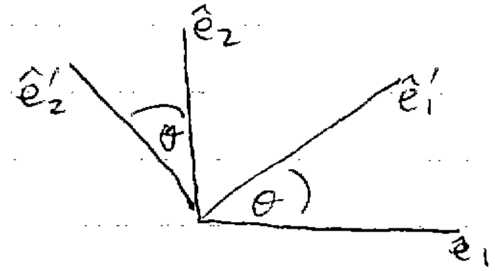
$$\vec{r}\,' = \overleftrightarrow{\lambda} \cdot \vec{r}$$

double arrow indicates a matrix

$$x_1' = \lambda_{11} x_1 + \lambda_{12} x_2 + \lambda_{13} x_3$$
$$x_2' = \lambda_{21} x_1 + \lambda_{22} x_2 + \lambda_{23} x_3$$
$$x_3' = \lambda_{31} x_1 + \lambda_{32} x_2 + \lambda_{33} x_3$$

$\overleftrightarrow{\lambda}$ is the rotation matrix.

in our 2D example



$$\lambda_{11} = (\hat{e}_1' \cdot \hat{e}_1) = \cos\theta$$
$$\lambda_{22} = (\hat{e}_2' \cdot \hat{e}_2) = \cos\theta$$
$$\lambda_{12} = (\hat{e}_1' \cdot \hat{e}_2) = \cos\left(\tfrac{\pi}{2} - \theta\right) = \sin\theta$$
$$\lambda_{21} = (\hat{e}_2' \cdot \hat{e}_1) = \cos\left(\tfrac{\pi}{2} + \theta\right) = -\sin\theta$$

Rotation matrix is $\overleftrightarrow{\lambda} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

In a 3D rotation matrix, not all the nine λ_{ij} are independent of each other. In fact, there are only 3 independent parameters.

To get the constraints that the λ_{ij} satisfy, consider

$\delta_{ij} = \hat{e}_i' \cdot \hat{e}_j'$

since $\hat{e}_i'$ are orthonormal basis vectors

we can write $\hat{e}_i' = (\hat{e}_i' \cdot \hat{e}_1)\hat{e}_1 + (\hat{e}_i' \cdot \hat{e}_2)\hat{e}_2 + (\hat{e}_i' \cdot \hat{e}_3)\hat{e}_3$

$$\Rightarrow \hat{e}_i' = \sum_k \lambda_{ik} \hat{e}_k$$

similarly $\hat{e}_j' = \sum_m \lambda_{jm} \hat{e}_m$

so $$\hat{e}_i' \cdot \hat{e}_j' = \sum_{km} \lambda_{ik} \lambda_{jm} (\hat{e}_k \cdot \hat{e}_m)$$

$$= \sum_{km} \lambda_{ik} \lambda_{jm} \delta_{km}$$

$$\hat{e}_i' \cdot \hat{e}_j' = \boxed{\sum_k \lambda_{ik} \lambda_{jk} = \delta_{ij}}$$

For $i=j$ we get $\lambda_{i1}^2+\lambda_{i2}^2+\lambda_{i3}^2=1$
this represents 3 equations correspondig to $i=1,2,3$
For $i\neq j$ we get

$$\lambda_{i1}\lambda_{j1}+\lambda_{i2}\lambda_{j2}+\lambda_{i3}\lambda_{j3}=0$$

this represents 3 equations corresponding to
$(i,j)=(1,3),(1,2),(2,3)$

(note $i=1, j=3$ gives same equation as $i=3, j=1$)

In total we get 6 equations for the nine λ_{ij}
$\Rightarrow$ there are 3 free parameters describing the rotation

One way to represent these three parameters is to say one has a rotation of angle θ about an axis oriented in direction $\hat{n}$. Since $\hat{n}$ is a unit vector, $n_1^2+n_2^2+n_3^2=1$, so giving the direction $\hat{n}$ uses 2 parameters, and giving the angle θ uses the 3rd parameter,

There are other ways to parameterize the rotation
(Euler angles used in solid body rotations)

Inverse rotation

suppose we know the coordinates (x_1', x_2', x_3') with respect to basis $\hat{e}_1', \hat{e}_2', \hat{e}_3'$. Then what are coordinates ~~with~~ (x_1, x_2, x_3) with respect to basis $\hat{e}_1, \hat{e}_2, \hat{e}_3$?

everything is the same as before ~~except prime~~ except prime $\Leftrightarrow$ unprimed

$$\vec{r} = \overleftrightarrow{\lambda}' \cdot \vec{r}' \quad \text{where} \quad \lambda_{ij}' = (\hat{e}_i \cdot \hat{e}_j') = \lambda_{ji}$$

So if
$$\overleftrightarrow{\lambda} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix}$$

then
$$\overleftrightarrow{\lambda}' = \begin{pmatrix} \lambda_{11} & \lambda_{21} & \lambda_{31} \\ \lambda_{12} & \lambda_{22} & \lambda_{32} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} \end{pmatrix}$$

we say that $\overleftrightarrow{\lambda}'$ is the transposed matrix of $\overleftrightarrow{\lambda}$

$$\overleftrightarrow{\lambda}' = \overleftrightarrow{\lambda}^t$$

the transpose of any matrix $\overleftrightarrow{A}$ with elements a_{ij} is the matrix $\overleftrightarrow{A}^t$ with elements

$$(A^t)_{ij} = a_{ji}$$

$(ij)^{th}$ element of $\overleftrightarrow{A}^t$ = $(ji)^{th}$ element of $\overleftrightarrow{A}$

Note also that $\overleftrightarrow{\lambda}'$ is the matrix of the inverse rotation of $\overleftrightarrow{\lambda}$. If one rotates by $\overleftrightarrow{\lambda}$ and then by $\overleftrightarrow{\lambda}'$, one winds up in the same original basis.

Algebraically: $\vec{r} = \overleftrightarrow{\lambda}' \cdot \vec{r}'$ but $\vec{r}' = \overleftrightarrow{\lambda} \cdot \vec{r}$

$$\text{So } \vec{r} = \overleftrightarrow{\lambda}' \cdot \vec{r}' = \overleftrightarrow{\lambda}' \cdot (\overleftrightarrow{\lambda} \cdot \vec{r})$$
$$= (\overleftrightarrow{\lambda}' \cdot \overleftrightarrow{\lambda}) \cdot \vec{r}$$

true for all $\vec{r}$

$$\Rightarrow (\overleftrightarrow{\lambda}' \cdot \overleftrightarrow{\lambda}) = \overleftrightarrow{I} \text{ identity matrix}$$

or in terms of indices $x_i = \sum_j \lambda'_{ij} x'_j$

and $x'_j = \sum_k \lambda_{jk} x_k$

$$\Rightarrow x_i = \sum_{jk} \lambda'_{ij} \lambda_{jk} x_k$$
$$= \sum_k \left(\sum_j \lambda'_{ij} \lambda_{jk} \right) x_k$$

above is true for all vectors $\vec{r}$

$$\Rightarrow \sum_j \lambda'_{ij} \lambda_{jk} = \delta_{ij}$$

the inverse of a matrix $\overleftrightarrow{A}$ is written as $\overleftrightarrow{A}^{-1}$,
$\overleftrightarrow{A}^{-1} \cdot \overleftrightarrow{A} = \overleftrightarrow{A} \cdot \overleftrightarrow{A}^{-1} = \overleftrightarrow{I}$

So we conclude that $\overleftrightarrow{\lambda}' = \overleftrightarrow{\lambda}^{-1}$

Combining with earlier result $\overleftrightarrow{\lambda}' = \overleftrightarrow{\lambda}^{t}$

we get the very important result for rotation matrices that

$$\boxed{\overleftrightarrow{\lambda}^{-1} = \overleftrightarrow{\lambda}^{t}}$$

inverse = transpose

Aside: assumed you understood matrix multiplication
$\overleftrightarrow{C} = \overleftrightarrow{A} \cdot \overleftrightarrow{B}$ means

$$C_{ij} = \sum_k A_{ik} B_{kj}$$

$$\cancel{C_{11} = A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31}}$$

Any matrix that satisfies the condition $A^{-1} = A^{t}$ is said to be an orthogonal matrix

Another property of orthogonal matrices is that if one takes their determinant, then $|\overleftrightarrow{A}|^2 = 1$

This follows from the following:
For any matrix $\overleftrightarrow{A}$, which has an inverse $\overleftrightarrow{A}^{-1}$

$$|\overleftrightarrow{A}^{-1}| = \frac{1}{|A|} \quad \text{so} \quad |A||A^{-1}| = 1$$

also, for any matrix $\overleftrightarrow{A}$, $|A| = |A^{t}|$. For orthogonal, $A^{-1} = A^{t}$, so $|A||A^{-1}| = |A||A^{t}| = |A|^2 = 1$

Another property of orthogonal matrices:

If A and B are orthogonal, so is ~~A~~ $C = AB$

Proof: $C^{-1} = (AB)^{-1} = B^{-1}A^{-1}$ see HW

$= B^t A^t$ since A, B orthogonal

$$(C^{-1})_{ij} = \sum_k (B^t)_{ik} (A^t)_{kj} = \sum_k B_{ki} A_{jk}$$

$$= \sum_k A_{jk} B_{ki} = C_{ji} = (C^t)_{ij}$$

So $C^{-1} = C^t$ and C is orthogonal!

Physically this means that two successive rotations about different axes can always be viewed as a single rotation about a ~~separate axis~~ some appropriate other axis.

Proper and Improper Rotations

For ~~a rotation~~ an orthogonal matrix $\overleftrightarrow{A}$, $|\overleftrightarrow{A}|^2 = 1$

If $|\overleftrightarrow{A}| = 1$ we say it is a proper rotation
$|\overleftrightarrow{A}| = -1$ we say it is an improper rotation.

Example of improper rotation is inversion

$$\overleftrightarrow{\lambda} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \hat{e}_i' = -\hat{e}_i$$

turns a right handed coordinate basis into a left handed coordinate basis.

All orthogonal matrices obtained by a series of ordinary rotations have determinant = 1.
All orthogonal matrices which are improper rotations can be written as a product of proper rotations times an inversion.

Under inversion, the components of a vector are reflected.

$$\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$$
$$= -A_1\hat{e}_1' - A_2\hat{e}_2' - A_3\hat{e}_3' \quad \leftarrow \text{inverted basis}$$

so $A_i' = -A_i$

Consider cross product $\vec{C} = \vec{A} \times \vec{B}$ under inversion

$$C_i = \sum_k \varepsilon_{ijk} A_j B_k$$

under inversion

$$C_i' = \sum_k \varepsilon_{ijk} A_j' B_k' = \sum_k \varepsilon_{ijk} (-A_j)(-B_k)$$

$$= \sum_k \varepsilon_{ijk} A_j B_k = C_i \quad !$$

$$C_i' = C_i$$

we say $\vec{C}$ is a psuedo vector - under an inversion its coordinates are <u>not</u> reflected.

The laws of physics must be written so that all terms in an equation are either vectors or psuedo vecto

Example: Lorenz force $\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$

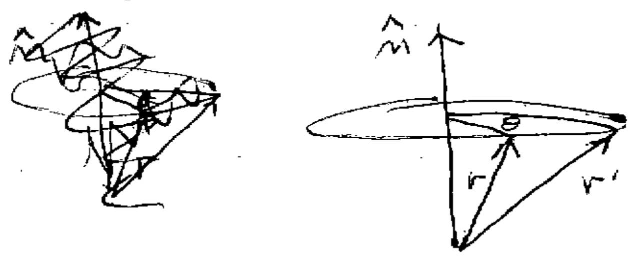
$\vec{F}$ and $\vec{v}$ are vectors $\rightarrow$ $\vec{E}$ is a vector

$\vec{B}$ is a psuedovector

(since vector × psuedo = vect

Finite and Infinitesimal Rotations

A rotation can be described by 3 parameters. Since a vector also has 3 parameters, one might think that a rotation can be represented by a vector. For example, we might try to represent a rotation through angle θ about the axis of rotation $\hat{n}$ by a vector $\vec{\theta} = \theta\hat{n}$



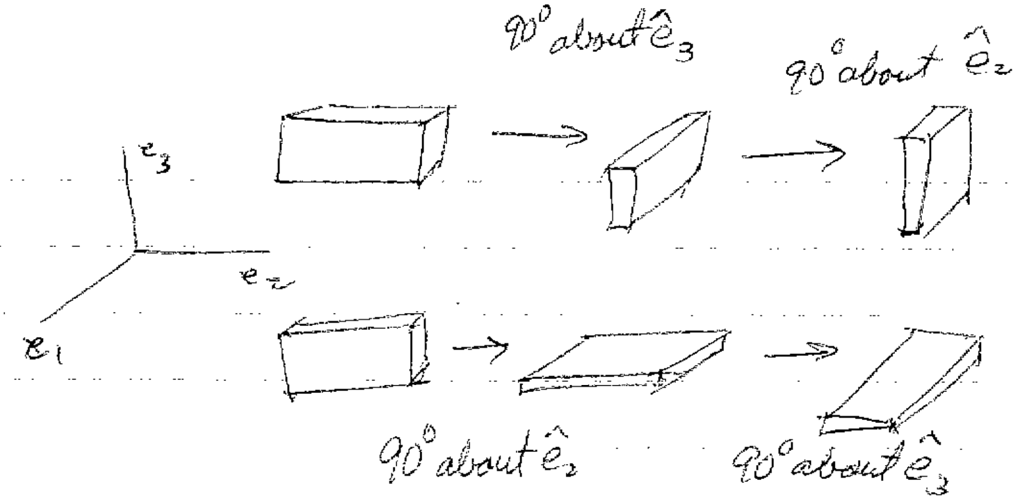
A second rotation through angle θ' about axis $\hat{n}'$ would be described by a vector $\vec{\theta}' = \theta'\hat{n}'$.

But if this idea of a rotation vector makes sense, one would want that a rotation $\vec{\theta}$, followed by a rotation $\vec{\theta}'$ would be described by the vector $\vec{\theta} + \vec{\theta}'$.

But since vector addition is commutative, $\vec{\theta} + \vec{\theta}' = \vec{\theta}' + \vec{\theta}$, this would imply that rotations commute.

But this cannot be so because rotations are given by matrices and matrices do not in general commute

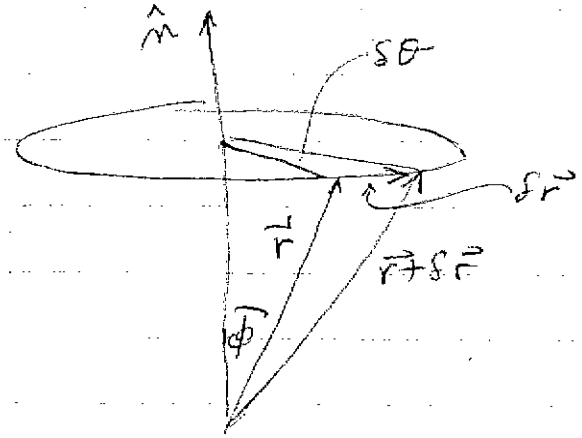
example: consider rotation by 90° about $\hat{e}_3$ followed by 90° rotation about $\hat{e}_2$. this is not the same as rotation by 90° about $\hat{e}_2$ followed by a 90° rotation about $\hat{e}_3$.



rotations do not commute.

However the above problem does not occur for infinitesmal rotations. Infinitesmal rotations do commute. This fact allows us to define the angular velocity vector $\vec{\omega}$.

Consider infinitesmal rotation through angle $\delta\theta$ about rotation axis $\hat{n}$, ~~and infinitesmal rotation~~



change in $\vec{r}$ in above rotation is
$|\delta\vec{r}| = r\sin\phi\,\delta\theta$
direction of $\delta\vec{r}$ is direction of $\hat{n}\times\vec{r}$

$$\Rightarrow \quad \delta\vec{r} = \delta\theta\,\hat{n}\times\vec{r} = \vec{\delta\theta}\times\vec{r}$$

where $\vec{\delta\theta} = \delta\theta\,\hat{n}$ is the vector describing the infinitesmal rotation.

Now consider a second rotation $\vec{\delta\theta}' = \delta\theta'\,\hat{n}'$

After the 1^{st} rotation, our vector $\vec{r}$ is transformed into the vector $\vec{r} + \delta\vec{\theta} \times \vec{r}$.
After the 2^{nd} rotation, the vector is transformed into

$$(\vec{r} + \delta\vec{\theta} \times \vec{r}) + \delta\vec{\theta}' \times (\vec{r} + \delta\vec{\theta} \times \vec{r})$$

$$= \vec{r} + \delta\vec{\theta} \times \vec{r} + \delta\vec{\theta}' \times \vec{r} + \delta\vec{\theta}' \times (\delta\vec{\theta} \times \vec{r})$$

2nd order in infinitesimals ⇒ ignore

So we wind up at vector

$$\vec{r} + \delta\vec{\theta} \times \vec{r} + \delta\vec{\theta}' \times \vec{r} = \vec{r} + (\delta\vec{\theta} + \delta\vec{\theta}') \times \vec{r}$$

If we now rotate by $\delta\vec{\theta}'$ first, we get $\vec{r} + \delta\vec{\theta}' \times \vec{r}$
Then if we next rotate by $\delta\vec{\theta}$ we get

$$(\vec{r} + \delta\vec{\theta}' \times \vec{r}) + \delta\vec{\theta} \times (\vec{r} + \delta\vec{\theta}' \times \vec{r})$$

$$= \vec{r} + \delta\vec{\theta}' \times \vec{r} + \delta\vec{\theta} \times \vec{r} + \delta\vec{\theta} \times (\delta\vec{\theta}' \times \vec{r})$$

2nd order so ignore

$$= \vec{r} + (\delta\vec{\theta}' + \delta\vec{\theta}) \times \vec{r}$$

So the two infinitesimal rotations do commute.
Moreover, the product of the two rotations looks just like a single infinitesimal rotation with rotation vector $\delta\vec{\theta}' + \delta\vec{\theta}$ as expected if the infinitesimal rotation is indeed describable as a vector.

For a rotating solid body, the change in position of a point at position $\vec{r}$, after an infinitesimal rotation by $\delta\vec{\theta}$ is

$$\delta\vec{r} = \delta\vec{\theta} \times \vec{r}$$

If this rotation takes place in a time δt, then we get for the velocity of the point at $\vec{r}$,

$$\vec{v} = \frac{\delta\vec{r}}{\delta t} = \frac{\delta\vec{\theta}}{\delta t} \times \vec{r}$$

We define the ~~angular~~ instantaneous angular velocity of the solid body as

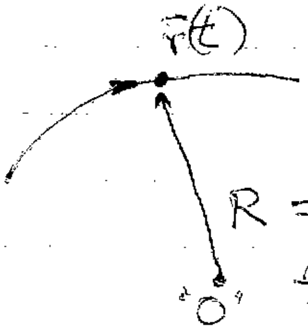
$$\vec{\omega} = \frac{\delta\vec{\theta}}{\delta t}$$

and then

$$\boxed{\vec{v} = \vec{\omega} \times \vec{r}}$$

the fact that $\delta\vec{\theta}$ is a vector $\Rightarrow$ angular velocity $\vec{\omega}$ is a vector - points along the instantaneous axis of rotation.

For a point particle traveling on a trajectory $\vec{r}(t)$, we can also define an instantaneous angular velocity as follows. Consider the plane spanned by $\vec{v}(t) = \dot{\vec{r}}(t)$ and $\vec{a}(t) = \ddot{\vec{r}}(t)$. ~~Locally, the particle~~ Instantaneously, the particle is moving in this plane, on a path with an instantaneous radius of curvature plane containg $\dot{\vec{r}}$ ad $\ddot{\vec{r}}$ at time t



R = radius of curvature at point $\vec{r}(t)$ in this plane

Instantaneously, the particle looks as if it is going in a circle of radius R, in the plane spanned by $\dot{\vec{r}}$ ad $\ddot{\vec{r}}$. An axis through point "O", perpendicular to this plane, is the instantaneous axis of rotation. If the particle moves an angular distance $d\theta$ in time dt, the instantaneous angular velocity is $\vec{\omega} = \frac{d\theta}{dt}\hat{n}$ where $\hat{n}$ points along the instantaneous axis of rotation.

Then $\quad \vec{v} = \vec{\omega} \times \vec{r}$