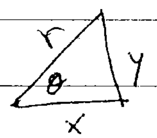
Conic Sections

$$\frac{\alpha}{r} = 1 + \varepsilon \cos\theta$$

to see the above does indeed describe conic sections we can translate into Cartesian coordinates



$$\Rightarrow \quad r = \sqrt{x^2+y^2}$$

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\alpha}{r} = 1 + \varepsilon\cos\theta \Rightarrow \frac{\alpha}{\sqrt{x^2+y^2}} = 1 + \frac{\varepsilon x}{\sqrt{x^2+y^2}}$$

$$\Rightarrow \quad \alpha - \varepsilon x = \sqrt{x^2+y^2}$$

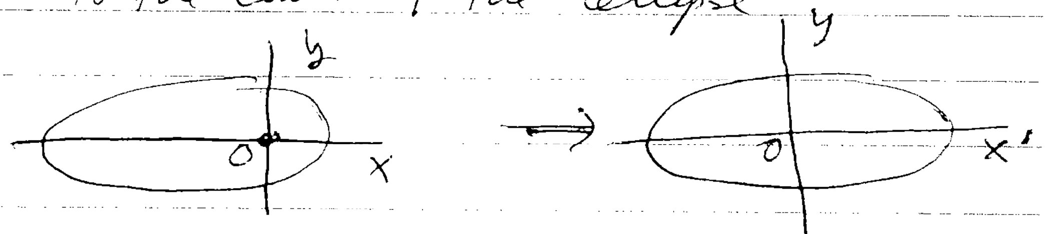
$$\alpha^2 + \varepsilon^2 x^2 - 2\alpha\varepsilon x = x^2 + y^2$$

$$x^2(1-\varepsilon^2) + 2\alpha\varepsilon x + y^2 = \alpha^2$$

shift x coordinate to complete the square

$$x' = x + x_0$$

For an ellipse, this translates origin from the focus to the center of the ellipse



origin at focus $\qquad\qquad$ origin at center substitute in $x' = x + x_0 \Rightarrow x = x' - x_0$

$$\Rightarrow (x'-x_0)^2(1-\varepsilon^2) + 2\alpha\varepsilon(x'-x_0) + y^2 = \alpha^2$$

$$x'^2(1-\varepsilon^2) + (2\alpha\varepsilon - 2x_0(1-\varepsilon^2))x' + y^2 = \alpha^2 - x_0^2(1-\varepsilon^2) + 2\alpha\varepsilon x_0$$

Choose x_0 to make the term linear in x' vanish

$$x_0 = \frac{\alpha\varepsilon}{1-\varepsilon^2}$$

Substitute in this value of x_0

$$x'^2(1-\varepsilon^2) + y^2 = \alpha^2 - \frac{\alpha^2\varepsilon^2}{(1-\varepsilon^2)^2}(1-\varepsilon^2) + 2\frac{\alpha^2\varepsilon^2}{1-\varepsilon^2}$$

$$= \alpha^2 - \frac{\alpha^2\varepsilon^2}{1-\varepsilon^2} + \frac{2\alpha^2\varepsilon^2}{1-\varepsilon^2}$$

$$= \alpha^2 + \frac{\alpha^2\varepsilon^2}{1-\varepsilon^2} = \alpha^2\left(1 + \frac{\varepsilon^2}{1-\varepsilon^2}\right)$$

$$x'^2(1-\varepsilon^2) + y^2 = \alpha^2\left(\frac{1}{1-\varepsilon^2}\right)$$

$$\Rightarrow \quad x'^2\frac{(1-\varepsilon^2)^2}{\alpha^2} + y^2\frac{(1-\varepsilon^2)}{\alpha^2} = 1$$

for $\varepsilon < 1$,
above is now in standard form of an ellipse

$$\frac{x'^2}{a^2} + \frac{y^2}{b^2} = 1$$

with major axis a and minor axis b

Comparing,

We have major axis $\boxed{a = \dfrac{\alpha}{1-\varepsilon^2}}$

minor axis $\boxed{b = \dfrac{\alpha}{\sqrt{1-\varepsilon^2}}}$

For $\varepsilon = 0$, $a = b$ and we have a circle

Note - the focus of the ellipse is at $x = 0$, or

$$x' = x_0 = \frac{\alpha \varepsilon}{1-\varepsilon^2} = a\varepsilon$$

For $\varepsilon > 1$, $(1-\varepsilon^2)^2 > 0$ but $1-\varepsilon^2 < 1$

Above equation becomes $\dfrac{x'^2}{a^2} - \dfrac{y^2}{b^2} = 1$

with $a = \dfrac{\alpha}{|1-\varepsilon^2|}$, $b = \dfrac{\alpha}{\sqrt{|1-\varepsilon^2|}}$

This is the equation of a <u>hyperbola</u>

For $\varepsilon = 1$, our earlier equation

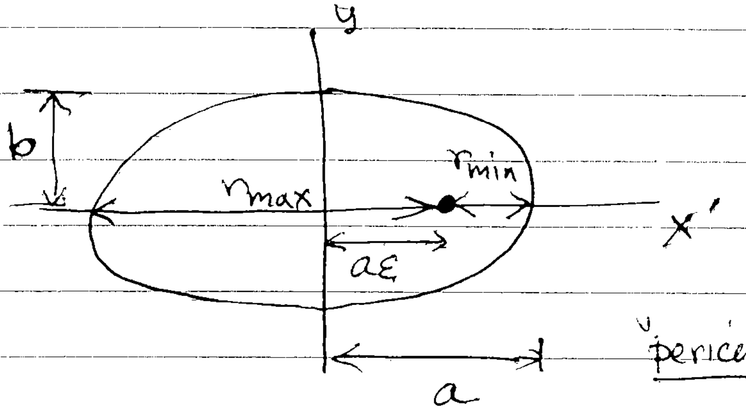
$$x^2(1-\varepsilon^2) + 2\alpha\varepsilon x + y^2 = \alpha^2$$

becomes $$2\alpha x + y^2 = \alpha^2$$

$$x = \frac{y^2 - \alpha^2}{2\alpha} = \frac{1}{2\alpha} y^2 - \frac{1}{2}\alpha$$

This is equation of a <u>parabola</u>

For elliptical orbits, $\varepsilon < 1$



$$\frac{\alpha}{r} = 1 + \varepsilon\cos\theta$$

$$r = \frac{\alpha}{1+\varepsilon\cos\theta}$$

"pericenter" $r_{min} = r(\theta = 0) = \dfrac{\alpha}{1+\varepsilon}$

"apocenter" $r_{max} = r(\theta = \pi) = \dfrac{\alpha}{1-\varepsilon}$

major axis $a = \dfrac{\alpha}{1-\varepsilon^2}$ (could also get this from $2a = r_{min} + r_{max}$)

minor axis $b = \dfrac{\alpha}{\sqrt{1-\varepsilon^2}}$

with $\alpha = \dfrac{l^2}{\mu k}$ and $\varepsilon = \sqrt{1 + \dfrac{2El^2}{\mu k^2}}$

we have

$$a = \frac{l^2}{\mu k}\frac{1}{\left(-\frac{2El^2}{\mu k^2}\right)} = \frac{k}{2|E|}$$

as $-E = |E|$ for $\varepsilon < 1$

similarly $b = \dfrac{l^2}{\mu k}\dfrac{1}{\sqrt{\left(\frac{2|E|l^2}{\mu k^2}\right)}} = \dfrac{l}{\sqrt{2\mu|E|}}$

$$\boxed{a = \frac{\alpha}{1-\varepsilon^2} = \frac{k}{2|E|}, \quad b = \frac{\alpha}{\sqrt{1-\varepsilon^2}} = \frac{l}{\sqrt{2\mu|E|}}}$$

period of elliptical orbit:

From Kepler's 2nd law, $\frac{dA}{dt} = \frac{1}{2}\frac{l}{\mu}$ ← dA is area swept out

$\Rightarrow dt = \frac{2\mu}{l} dA$

$\Rightarrow$ period $T = \frac{2\mu}{l} A$ $\quad$ A = area of ellipse

The area of an ellipse is $A = \pi ab$
where a, b are the major + ~~semimajor~~ minor axes.

$$T = \frac{2\mu}{l}\pi\left(\frac{k}{2|E|}\right)\left(\frac{l}{\sqrt{2\mu|E|}}\right) = \pi k\sqrt{\frac{\mu}{2}}\frac{1}{|E|^{3/2}}$$

but also $T = \frac{2\mu}{l}\pi ab = \frac{2\mu}{l}\pi a^2\sqrt{1-\varepsilon^2}$

but from $a = \frac{\alpha}{1-\varepsilon^2}$ we get $\sqrt{1-\varepsilon^2} = \sqrt{\frac{\alpha}{a}} = \frac{l}{\sqrt{\mu k}}\frac{1}{\sqrt{a}}$

$$T = \frac{2\mu}{l}\pi a^{3/2}\frac{l}{\sqrt{\mu k}}$$

$$\Rightarrow T^2 = \frac{4\mu^2\pi^2 a^3}{\mu k} = \boxed{\frac{4\pi^2\mu}{k}a^3 = T^2}$$

Keppler's 3rd law

To conclude, we have derived Kepler's 3 Laws:

1) Each planet moves in an elliptical orbit with the sun at one focus

2) The radius vector drawn from the sun to a planet sweeps out equal areas in equal amounts of time, ie $\frac{dA}{dt} = \frac{1}{2}\frac{l}{\mu}$ is constant

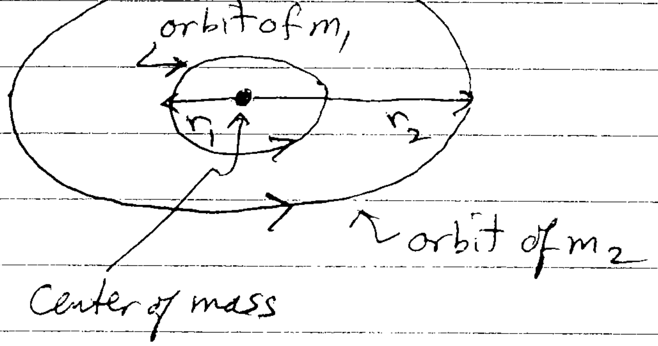
3) The square of the period of revolution about the sun is proportional to the cube of the major axis of the orbit, ie

$$T^2 = \frac{4\pi^2 \mu}{k} a^3$$

It was Newton's efforts to derive Kepler's laws that led him to develop his 3 laws of mechanics

Keppler's 3rd Law $T^2 = \frac{4\pi^2 \cancel{\mu}}{k} a^3$

applies to orbit of the effective one body seperation vector $\vec{r} = \vec{r}_1 - \vec{r}_2$, NOT to the physical orbits of m_1 and m_2



whereas Keppler applied it directly to the orbit of planets about the sun, ie "a" he took as major axis of elliptical orbit around sun, and he found that T^2/a^3 was constant, independent of the particular planet.

How is this true?

① For a planet of mass m, orbiting the sun of mass M, because $m \ll M$, the center of mass is so close to the sun, that we can ignore the difference between the distance of the planet from the sun, and its distance from the center of mass. If r_1 is position of sun relative to CM, and r_2 is position of planet relative to CM, then

$$r_1 = \frac{m}{m+M} r \approx \frac{m}{M} r \ll r$$

$$r_2 = \frac{M}{m+M} r \approx \frac{M}{M} r = r$$

So $r_2 \approx r$ and major axis a is

essentially the same as furthest distance of planet from the sun (rather than from CM)

(2) Since $k = GmM$ and $\mu = \frac{mM}{m+M}$

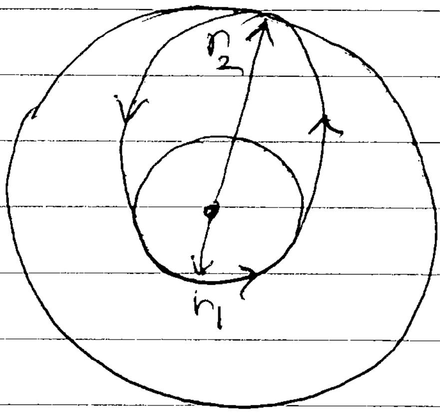
we have $\frac{\mu}{k} = \frac{mM}{m+M} \frac{1}{GmM} = \frac{1}{G(m+M)} \approx \frac{1}{GM}$

as $m \ll M$

So $T^2 \approx \frac{4\pi^2}{GM} a^3$

$\therefore \frac{T^2}{a^3} \approx \frac{4\pi^2}{GM}$ is same constant for each planet

Orbital dynamics



to travel to Mars with minimum energy expenditure,

1. start in circular earth orbit at r_1
2. increase speed to go into elliptical orbit with major axis $2a = r_1 + r_2$
3. then decrease speed to put into circular orbit at r_2

What changes in speed are needed at steps 2 and 3?

For circles and ellipses, we have for the major axis

$$a = \frac{k}{2|E|} = \frac{k}{-2E} \Rightarrow E = -\frac{k}{2a}$$

For earth orbit at radius $a = r_1$

$$\underset{\substack{\uparrow \\ \text{total energy}}}{E} = -\frac{k}{2r_1} = \underset{\substack{\uparrow \\ \text{kinetic energy}}}{\tfrac{1}{2}mv_1^2} - \underset{\substack{\uparrow \\ \text{potential energy}}}{\frac{k}{r_1}}$$

$$\Rightarrow \tfrac{1}{2}mv_1^2 = \frac{k}{2r_1}$$

$$v_1 = \sqrt{\frac{k}{mr_1}}$$ speed in orbit at radius r_1

For the elliptical orbit with $r_{min} = r_1$ and $r_{max} = r_2$

major axis $a = \frac{r_{min} + r_{max}}{2} = \frac{r_1 + r_2}{2}$

$$E = \frac{-k}{2a} = \frac{-k}{r_1 + r_2} = \underbrace{\frac{1}{2} m v_{t1}^2}_{\substack{\text{kinetic energy} \\ \text{at } r_1 \\ \text{on ellipse}}} - \underbrace{\frac{k}{r_1}}_{\substack{\text{potential energy} \\ \text{at } r_1 \\ \text{on ellipse}}}$$

v_{t1} is the speed we need to be in the elliptical orbit at $r_{min} = r_1$

$$\frac{1}{2} m v_{t1}^2 = k \left(\frac{1}{r_1} - \frac{1}{r_1 + r_2} \right) = k \left(\frac{r_2}{r_1 + r_2} \right)$$

$$v_{t1} = \sqrt{\frac{2k}{m} \left(\frac{r_2}{r_1 + r_2} \right)}$$

So to go from earth orbit into the transfer ellipse we need to increase speed an amount

$$\Delta v = v_{t1} - v_1 = \sqrt{\frac{k}{m r_1}} - \sqrt{\frac{2k}{m} \left(\frac{r_2}{r_1 + r_2} \right)}$$

Similarly, to be in mars orbit at r_2 we need speed

$$v_2 = \sqrt{\frac{k}{m r_2}}$$

In the transfer elliptical orbit, at $r_{max} = r_2$ we have speed

$$v_{t2} = \sqrt{\frac{2k}{m} \left(\frac{r_1}{r_1 + r_2} \right)}$$

So to go from transfer ellipse into mars orbit, we need to decrease speed an amount

$$\Delta v = v_{t2} - v_2 = \sqrt{\frac{2k}{m}\left(\frac{r_1}{r_1+r_2}\right)} - \sqrt{\frac{k}{m r_2}}$$

The time to make this transfer is one half the period of the elliptical orbit

$$\text{time} = \frac{T}{2} = \pi\sqrt{\frac{m}{k}}\, a^{3/2} = \pi\sqrt{\frac{m}{k}}\left(\frac{r_1+r_2}{2}\right)^{3/2}$$

$$= 259 \text{ days}$$ to transfer from earth to mars orbit.