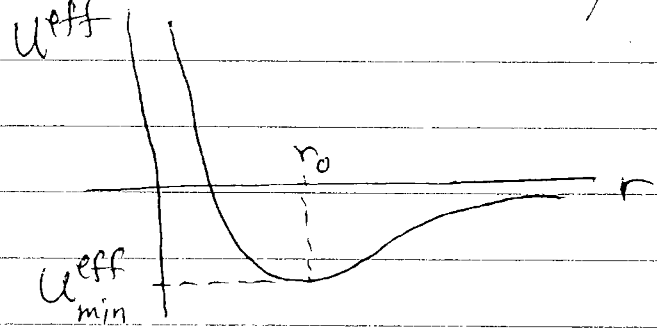
Stability of circular orbits

$$U_{eff}(r) = U(r) + \frac{\ell^2}{2\mu r^2}$$



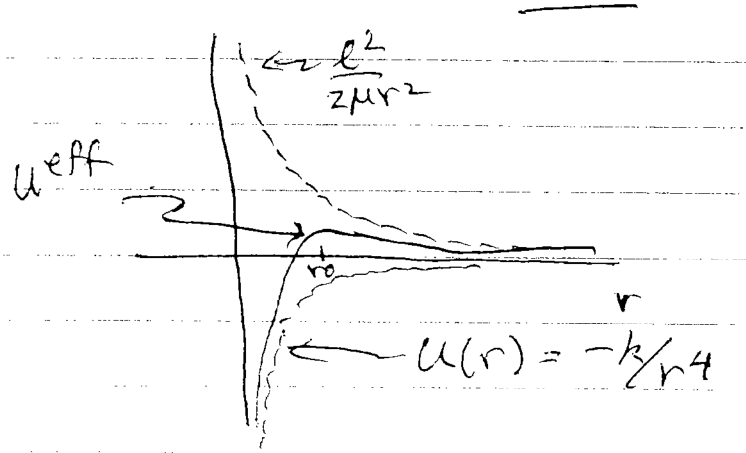
← for example when $U(r) = -\frac{k}{r}$

There will be a stable ~~circ~~ circular orbit at $r = r_0$

if $\left.\frac{dU^{eff}}{dr}\right|_{r=r_0} = 0$ (ie U^{eff} is extremal at r_0)

and $\left.\frac{d^2U^{eff}}{dr^2}\right|_{r=r_0} > 0$ (ie r_0 is a minimum)

We could have unstable orbits



If $|U(r)|$ decreases faster than $\frac{\ell^2}{2\mu r^2}$

we get a maximum in $U^{eff}(r)$.

Here $\left.\frac{dU^{eff}}{dr}\right|_{r=r_0} = 0$

but $\frac{d^2U^{eff}}{dr^2} < 0$ unstable

The two conditions for stable orbit at r_0 are therefore

① $$\left.\frac{dU^{eff}}{dr}\right|_{r_0} = \left.\frac{dU}{dr}\right|_{r_0} - \frac{\ell^2}{\mu r_0^3} = 0$$

since $F = -\frac{dU}{dr}$ $\Rightarrow$ $F = \frac{-\ell^2}{\mu r_0^3}$

② $$\left.\frac{d^2U^{eff}}{dr^2}\right|_{r_0} = \left.\frac{d^2U}{dr^2}\right|_{r_0} + \frac{3\ell^2}{\mu r_0^4} > 0$$

or $$-\frac{dF}{dr} + \frac{3\ell^2}{\mu r_0^4} > 0$$

For a power law force
$$F = \frac{-k}{r^n}$$

① $\Rightarrow$ $$-\frac{k}{r_0^n} = \frac{-\ell^2}{\mu r_0^3} \Rightarrow r_0^{n-3} = \frac{\mu k}{\ell^2}$$

② $\Rightarrow$ $$-\frac{nk}{r_0^{n+1}} + \frac{3\ell^2}{\mu r_0^4} > 0$$

$$\Rightarrow \frac{-nk}{r_0^{n-3}} + \frac{3\ell^2}{\mu} > 0$$

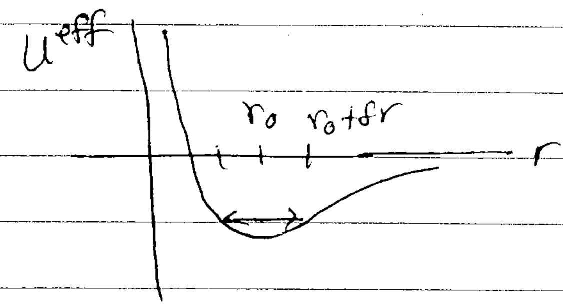
$$\Rightarrow \frac{-nk}{(\mu k/\ell^2)} + \frac{3\ell^2}{\mu} = \frac{\ell^2}{\mu}(3-n) > 0$$

$\Rightarrow (3-n) > 0 \Rightarrow \boxed{n < 3}$ for stable circular orbit

$n = 2$ for gravity

prob 8-35

Suppose we have a stable periodic orbit, and the orbit radius is perturbed to $r_0 + \delta r$



The particle will oscillate between radii $r_0 + \delta r$ and $r_0 - \delta r$ with frequency of oscillation

$$\omega_0^2 = \left[\left(\frac{d^2 U^{eff}}{dr^2} \right)_{r=r_0} \frac{1}{\mu} \right] = \frac{\text{curvature}}{\text{mass}}$$

From previous page, for $F = -\frac{k}{r^n}$

$$\left. \frac{d^2 U^{eff}}{dr^2} \right|_{r_0} = \frac{\ell^2}{\mu r_0^4} (3-n)$$

$$\omega_0 = \sqrt{\frac{\ell^2}{\mu^2 r_0^4}(3-n)} = \frac{\ell}{\mu r_0^2} \sqrt{3-n}$$

since $\ell = \mu r_0^2 \dot{\theta}$

$$\omega_0 = \dot{\theta} \sqrt{3-n}$$

for gravitational force, $n=2$

$$\Rightarrow \omega_0 = \dot{\theta}$$

period of oscillation between r_{min} and r_{max} is just period of revolution of elliptical orbit, as expected

More generally,

$$\omega_0 = \dot{\theta}\sqrt{3-n}$$

$\Rightarrow$ period of oscillation $T = \dfrac{2\pi}{\omega_0} = \dfrac{2\pi}{\dot{\theta}\sqrt{3-n}}$

$$\Rightarrow \frac{d\theta}{dt} = \frac{2\pi}{T\sqrt{3-n}}, \Rightarrow \int_{\theta_1}^{\theta_2} d\theta = \int_0^T dt \, \frac{2\pi}{T\sqrt{3-n}}$$

$\Rightarrow$ change in angle of rotation $\Delta\theta$, after one period T, is

$$\Delta\theta = \frac{2\pi}{\sqrt{3-n}}$$

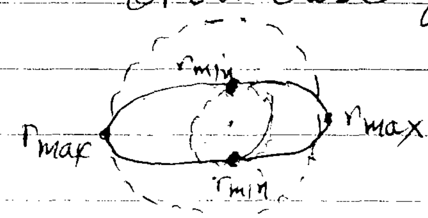
for $n = 2$, $\Delta\theta = 2\pi$, one complete revolution of origin in each period of oscillation of r.

$n = 1$, $\Delta\theta = \dfrac{2\pi}{\sqrt{2}}$, less than one complete revolution of origin in time T. $\dfrac{\Delta\theta}{2\pi}$ is irrational $\Rightarrow$ orbit is not closed

$n = -1$ $\Delta\theta = \dfrac{2\pi}{2} = \pi$, one half revolution each period, orbit closes after two periods

$F = -kr$

harmonic force



example: screened Coulomb potential

$$U(r) = -\frac{k}{r} e^{-r/a} \quad \sim \frac{-1}{r} \text{ for } r \ll a$$
$$\sim e^{-r/a} \text{ for } r \gg a$$

expect ~~effect~~ that, depending on values of a and ℓ, we might have stable or unstable circular orbit
expect stable orbit when $r_0 \lesssim a$, unstable when $r_0 \gtrsim a$

$$U^{eff} = U(r) + \frac{\ell^2}{2\mu r^2} = -\frac{k}{r} e^{-r/a} + \frac{\ell^2}{2\mu r^2}$$

(1) $$\frac{dU^{eff}}{dr} = \left(\frac{k}{r^2} + \frac{k}{ar}\right) e^{-r/a} - \frac{\ell^2}{\mu r^3} = 0$$

(2) $$\frac{d^2 U^{eff}}{dr^2} = \left(\frac{-2k}{r^3} - \frac{k}{ar^2} - \frac{k}{a^2 r} - \frac{k}{a^2 r}\right) e^{-r/a} + \frac{3\ell^2}{\mu r^4} > 0$$

$$= -k\left(\frac{2}{r^3} + \frac{2}{ar^2} + \frac{1}{a^2 r}\right) e^{-r/a} + \frac{3\ell^2}{\mu r^4} > 0$$

from (1): $$\left(kr + \frac{kr^2}{a}\right) e^{-r/a} = \frac{\ell^2}{\mu}$$

solve for r to get radius of orbit as function of angular momentum ℓ. - no analytical solution in general - need to do numerically.

But still we can write $$e^{-r/a} = \frac{\ell^2}{\mu k} \frac{1}{(r + r^2/a)}$$

substitute into (2)

$$-k\left(\frac{2}{r^3}+\frac{2}{ar^2}+\frac{1}{a^2 r}\right)\frac{\ell^2}{\mu k}\frac{1}{(r+r^2/a)}+\frac{3\ell^2}{\mu r^4}>0$$

multiply by $\frac{\mu r^4}{\ell^2}$

$$\Rightarrow 3-\frac{\left(2r+2\frac{r^2}{a}+\frac{r^3}{a^2}\right)}{r+r^2/a}>0$$

$$\Rightarrow 3-\frac{2+2(r/a)+(r/a)^2}{1+(r/a)}>0$$

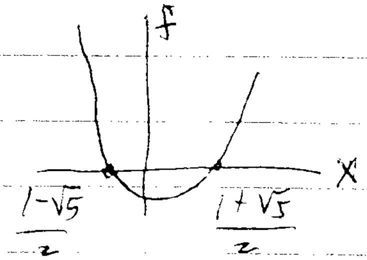
$$\Rightarrow 3+\frac{3r}{a}-2-2\left(\frac{r}{a}\right)-\left(\frac{r}{a}\right)^2>0$$

$$\Rightarrow 1+\frac{r}{a}-\left(\frac{r}{a}\right)^2>0$$

or $\left(\frac{r}{a}\right)^2-\frac{r}{a}-1<0$

define $x=\frac{r}{a}$, $f(x)\equiv x^2-x-1<0$

$f(x)$ has zeros at $x_0=\frac{1}{2}\pm\sqrt{\frac{1}{4}+1}=\frac{1}{2}\pm\sqrt{\frac{5}{4}}$
$=\frac{1\pm\sqrt{5}}{2}$



So $f(x)$ is negative between the two roots. Since $x=r/a$ must be positive, we have ~~phy~~ physical solution for all $0<\frac{r}{a}<\frac{1+\sqrt{5}}{2}$

or $r/a<1.618$ ← for all r in this range, there is one ℓ that gives stable orbit

If we compare the screened Coulomb potential
$$U^{sc}(r) = -\frac{k}{r} e^{-r/a}$$
to the ordinary Coulomb (or gravitation) potential
$$U^{c}(r) = -\frac{k}{r}$$

We found that for U^c, there is always a stable circular orbit for any angular momentum l. This orbit is at radius
$$r_0 = \frac{l^2}{\mu k}$$
as l increases, the radius r_0 increases, and can in principle become arbitrarily large – one never "escapes" the Coulomb force – there can be bound states at arbitrarily large radius.

For U^{sc} however, there are stable circular orbits only out to a distance $r_{max} = 1.618\,a$, set by the "screening length" a. No stable circular orbits exist for radius $r > r_{max}$. The relation between the radius r_0 ~~and~~ of the stable circular orbit and the angular momentum l, is given by the solution to
$$kr\left(1 + \frac{r}{a}\right)e^{-r/a} = \frac{l^2}{\mu}$$
Although we have no analytic solution, it is clear that as l increases, r_0 increases. Therefore there is a maximum angular momentum l_{max} for a stable circular orbit.

When one treats such problems quantum mechanically, one finds that angular momentum is quantized into discrete values

$$\ell^2 = p(1+p) \quad \text{for integer } p.$$

The conclusion for U^c, that there is a stable circular orbit for any value of ℓ no matter how big, translates into the quantum mechanical conclusion that there are ~~an infinite~~ now a countably infinite number of bound states in a Coulomb potential. – ie for the hydrogen atom, the principle quantum number n can be arbitrarily large.

The conclusion for U^{sc} that there are stable circular orbits only out to a particular ~~val~~ maximum value ℓ_{max} becomes the quantum mechanical conclusion that the screened Coulomb potential has only a finite number of bound states. Depending on the value of $\mu k a$, there might be, for example, only one, or even none, bound states. The larger $\mu k a$, the more bound states.

$$\text{Since} \quad \left(\frac{r}{a}\right)\left(1+\left(\frac{r}{a}\right)\right)e^{-(r/a)} = \frac{\ell^2}{\mu k a}$$

$\Rightarrow$ $\mu k a$ sets the scale of the max value of ℓ.

Consider the equation that ~~det~~ determines the extremal values of U^{eff}

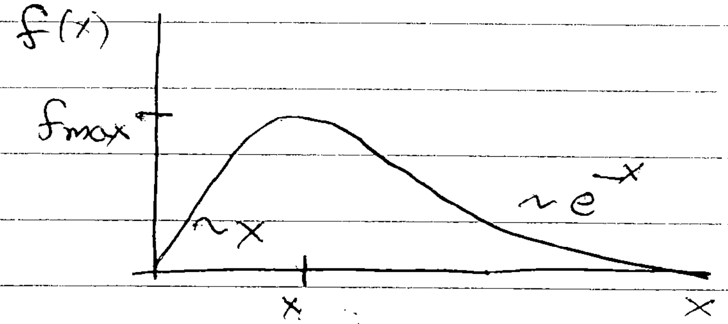
$$\left(\frac{r}{a}\right)\left(1+\frac{r}{a}\right)e^{-r/a} = \frac{\ell^2}{\mu k a}$$

let $x = \frac{r}{a}$, $y = \frac{\ell^2}{\mu k a}$

$$f(x) = x(1+x)e^{-x}$$

For given value of y, there are extrema of U^{eff} at x such that

$$f(x) = y$$



to determine x_0

$$\frac{df}{dx} = 0$$

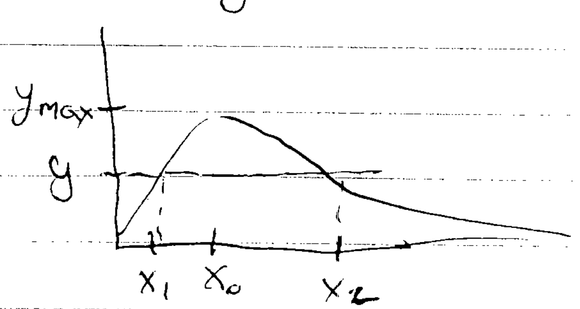
$$\Rightarrow \left(1+x+x - x(1+x)\right)e^{-x} = 0$$

$$\Rightarrow 1+x-x^2 = 0$$

$$\Rightarrow x_0 = \frac{1+\sqrt{5}}{2} = 1.618$$

let $y_{max} \equiv f(x_0)$

For $y < y_{max}$, there are two solutions to $f(x) = y$.



only x_1 is stable since

$$x_2 = \frac{r_2}{a} > 1.618 = x_0$$

as y increases above y_{max} there are no solutions

$$y_{max} = \frac{l_{max}^2}{\mu ka} = f(1.618) \quad \text{determines } l_{max}$$

for $l > l_{max}$ there are no stable solutions

$$l_{max}^2 = \mu ka\, f(1.618)$$

By QM, there is a minimum possible (non-zero) value to $l_{min}^2 = \hbar^2$

if $l_{min} > l_{max}$ there will be no bound states

if $\hbar^2 > \mu ka\, f(1.618)$ no bound states

$\Rightarrow$ decrease # bound states by decreasing parameter μka

as μka decreases, l_{max}^2 decreases
therefore fewer values of p such that
$l^2 = p(1+p)\hbar^2 < l_{max}^2$
ie fewer bound states