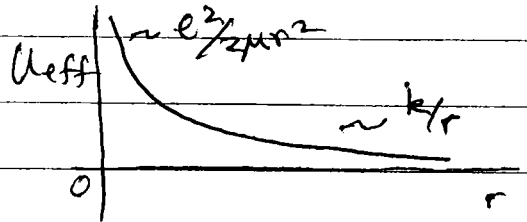# Rutherford Scattering

Repulsive Coulomb potential $U = \frac{k}{r}$

all orbits are unbounded

$$U^{eff} = \frac{k}{r} + \frac{\ell^2}{2\mu r^2}$$

general result

U_{eff} ~ $\frac{\ell^2}{2\mu r^2}$

~ $\frac{k}{r}$

0 r

$$\theta = \int \frac{\ell \, dr}{r^2 \sqrt{2\mu(E-U) - \frac{\ell^2}{2\mu r^2}}} + const$$

$$= \int \frac{\ell \, dr}{r^2 \sqrt{2\mu E - \frac{2\mu k}{r} - \frac{\ell^2}{r^2}}} + const$$

as before $u = \ell/r \implies du = -\frac{\ell}{r^2} dr$

$$\theta = -\int du \frac{1}{\sqrt{2\mu E - \frac{2\mu k}{\ell} u - u^2}} + const$$

substitute $v = u + \frac{\mu k}{\ell}$, $dv = du$

↑ this sign is opposite to what it was earlier for attractive $1/r^2$ potential

$$v^2 = u^2 + \frac{2\mu k u}{\ell} + \frac{\mu^2 k^2}{\ell^2}$$

$$\theta = -\int \frac{dv}{\sqrt{2\mu E + \frac{\mu^2 k^2}{\ell^2} - v^2}} + const$$

from this point on it is the same as for the attractive Coulomb potential — do the trigonometric substitution to do the integral and get

$$\Rightarrow \cos(\theta - \theta_0) = \frac{v}{c}$$

where θ_0 is the const of integratz

and $c = \sqrt{2\mu E + \frac{\mu^2 k^2}{\ell^2}}$

substitute back in terms of u

$$\cos(\theta - \theta_0) = \frac{u + \frac{\mu k}{\ell}}{\sqrt{2\mu E + \frac{\mu^2 k^2}{\ell^2}}} = \frac{\frac{\ell u}{\mu k} + 1}{\sqrt{1 + \frac{2E\ell^2}{\mu k^2}}}$$

substitute back $u = \ell/r$, and define

$$\alpha = \frac{\ell^2}{\mu k} \quad \text{and} \quad \varepsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}} \quad \text{as before}$$

because $U^{eff} > 0$,

$$\Rightarrow \cos(\theta - \theta_0) = \frac{\frac{\alpha}{r} + 1}{\varepsilon} \qquad \Rightarrow E > 0, \quad \varepsilon > 1$$

$$\frac{\alpha}{r} = -1 + \varepsilon \cos(\theta - \theta_0)$$

if choose

smallest value of r is when $\theta - \theta_0 = 0$. So ~~H~~

$\theta_0 = 0$, then $\theta = 0$ gives closest approach to origin.

above is equation for hyperbola

$$\frac{\alpha}{r} = -1 + \varepsilon \cos\theta$$

minus sign here is only difference
from attractive case studied earlier

$$\frac{\alpha}{r} = -1 + \varepsilon \cos\theta$$
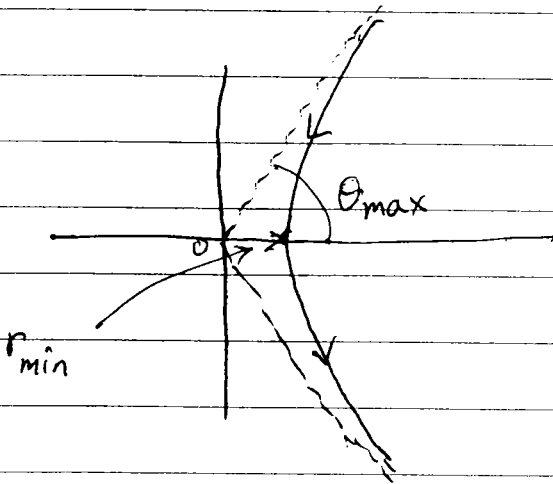
$r \to \infty$ when $-1 + \varepsilon \cos\theta = $

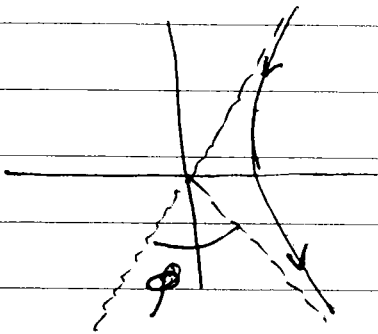closest approach to origin is

$$r_{min} = \frac{\alpha}{\varepsilon - 1}$$

Since $r \geq 0$, θ must always be in the range $|\theta| < \theta_{max}$ where $-1 + \varepsilon \cos\theta_{max} = 0$

$$\Rightarrow \cos\theta_{max} = \frac{1}{\varepsilon}$$

It is customary to define the scattering angle φ as follows. φ gives the angle of deflection away from the incoming direction.

$$\varphi = \pi - 2\theta_{max}$$
$$= \pi - 2\arccos\left(\frac{1}{\varepsilon}\right)$$
$$= \pi - 2\arccos\left(\frac{1}{\sqrt{1 + \frac{2E\ell^2}{Mk^2}}}\right)$$

scattering angle depends on energy E and angular momentum ℓ of the incoming particle

Alternatively, measuring the scattering angle φ lets one measure the combination $E\ell^2$

$$\frac{1}{\sqrt{1 + \frac{2E\ell^2}{Mk^2}}} = \cos\left(\frac{\pi - \varphi}{2}\right) = \sin\left(\frac{\varphi}{2}\right)$$
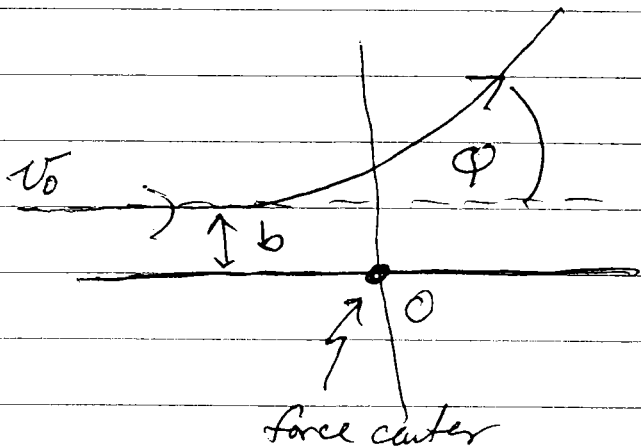
$$\Rightarrow \quad 1 + \frac{2E\ell^2}{\mu k^2} = \frac{1}{\sin^2\left(\frac{\varphi}{2}\right)}$$

$$\frac{2E\ell^2}{\mu k^2} = \frac{1}{\sin^2\left(\frac{\varphi}{2}\right)} - 1 = \frac{1 - \sin^2\left(\varphi/2\right)}{\sin^2\left(\varphi/2\right)}$$

$$= \frac{\cos^2\left(\varphi/2\right)}{\sin^2\left(\varphi/2\right)}$$

$$\Rightarrow \quad \frac{2E\ell^2}{\mu k^2} = \cot^2\left(\varphi/2\right)$$

One usually parameterizes the angular momentum by the _impact parameter_ b.

incident velocity v_0 approaching target at a distance b.

$$\ell = \mu v_0 b \quad \left\} \begin{array}{l} \text{initially} \\ \text{at } r \to \infty \end{array} \right.$$
$$E = \tfrac{1}{2}\mu v_0^2$$

force center

$$\Rightarrow \quad v_0 = \sqrt{\frac{2E}{\mu}}, \quad \ell = \mu b \sqrt{\frac{2E}{\mu}}$$

$$\ell = b\sqrt{2\mu E}$$

substitute into above

$$\frac{2El^2}{\mu k^2} = \frac{2Eb^2 2\mu E}{\mu k^2} = \frac{4E^2 b^2}{k^2} = \cot^2\left(\frac{\varphi}{2}\right)$$
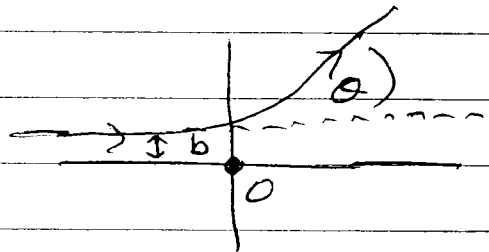
$$\Rightarrow \quad \frac{2Eb}{k} = \cot\left(\frac{\varphi}{2}\right)$$

$$\Rightarrow \quad \boxed{b = \frac{k}{2E} \cot\left(\frac{\varphi}{2}\right)}$$

relation between impact parameter b and scattering angle φ for particle of energy E

It is customary in scattering theory to call the angle φ above "θ". We now make this change — don't confuse this θ with our original θ which parameterized the hyperbolic trajectory

$$b = \frac{k}{2E} \cot\left(\frac{\theta}{2}\right)$$

Now consider a beam of incident particles of all different values of b.

radius b
thickness db

screen at radius R away from target.
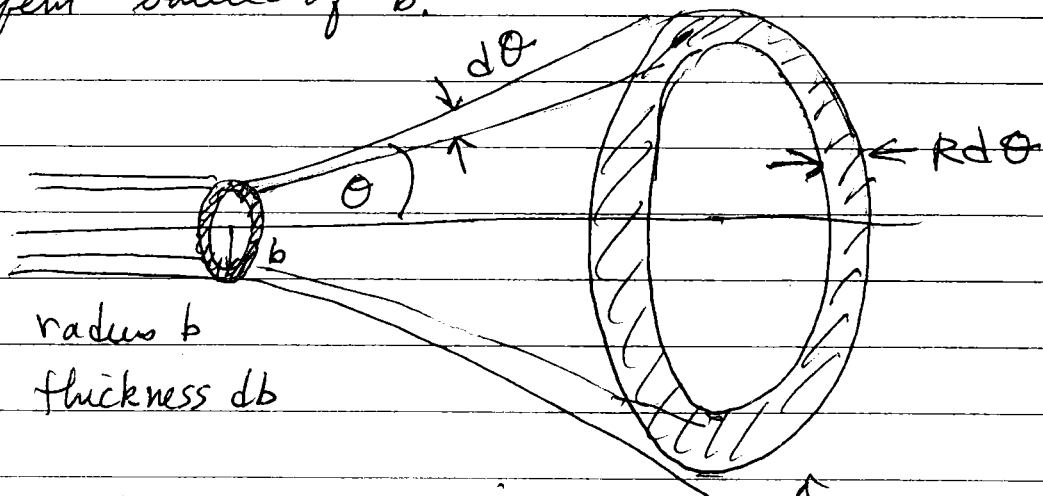
All incident particles with impact parameters between b and $b+db$, get scattered into angles between θ and $\theta + d\theta$

If the initial intensity of particles per unit area, per unit time is n_I, then the number of particles with impact parameter between b and $b+db$ is per unit time

$$n_I \, 2\pi b \, db.$$

These get scattered into an area on the screen of size $2\pi R^2 \sin\theta \, d\theta$. Since the number of scattered particles is conserved, the intensity $n(\theta)$ of scattered particles hitting the screen between θ and $\theta + d\theta$ is given by

$$n_I \, 2\pi b \, db = n(\theta) \, 2\pi R^2 \sin\theta \, d\theta$$

$$\Rightarrow \quad \frac{n(\theta)R^2}{n_I} = \frac{b\,db}{\sin\theta\,d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

$\dfrac{n(\theta)R^2}{n_I} \equiv \dfrac{d\sigma}{d\Omega}$ is called the <u>differential</u> <u>cross section</u>. It has <u>units of area</u>

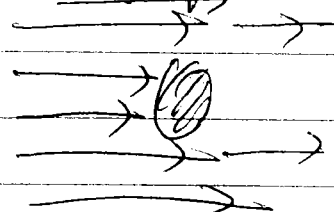$d\Omega \equiv 2\pi\sin\theta\,d\theta$ is called the ~~solid~~ <u>differential</u> <u>solid angle</u>

The total number of scattered particles per unit time is

$$N^{scatt} = \int n_I\,2\pi b\,db = \int_0^{\pi} n(\theta)\,2\pi R^2\sin\theta\,d\theta$$

$$= \int n_I\,\frac{d\sigma}{d\Omega}\,d\Omega = n_I\,\sigma$$

$\begin{array}{l}\text{total} \\ \text{cross section}\end{array} \qquad \sigma = \dfrac{N^{scatt}}{n_I}$

If we regarded the target as a fixed area a, such that a particle is scattered only if it hits the area, then the ~~number~~ number of scattered particles would be $N^{scatt} = n_I\,a$

Therefore the total cross section σ plays the role of the effective area for a more general scattering interaction. $\frac{d\sigma}{d\Omega}$ gives the fraction of the total scattering that goes into the solid angle $d\Omega$.

For the Coulomb interaction

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \qquad \text{with} \qquad b = \frac{k}{2E}\cot\left(\frac{\theta}{2}\right)$$

$$\frac{db}{d\theta} = \frac{k}{2E}\left(\frac{-1}{2}\right)\frac{1}{\sin^2\frac{\theta}{2}}$$

$$\frac{d\sigma}{d\Omega} = \frac{k}{2E}\frac{\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}\frac{1}{\sin\theta}\frac{k}{2E}\frac{1}{2}\frac{1}{\sin^2\frac{\theta}{2}}$$

use $\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{4}\left(\frac{k}{2E}\right)^2\frac{1}{\sin^4\left(\frac{\theta}{2}\right)}}$$ Rutherford scattering formula

for Coulomb potential scattering projectile of charge $Z_p e$ off target of charge $Z_T e$,

$$k = \frac{Z_p Z_T e^2}{4\pi\varepsilon_0}$$

<u>Note</u>: all the above analysis is with respect the difference coordinate $\vec{r} = \vec{r}_1 - \vec{r}_2$ between the projectile and the target. We have ignored the center of mass $\vec{R} = \frac{m_1}{M} \vec{r}_1 + \frac{m_2}{M} \vec{r}_2$ motion. Thus all our results, particularly the energy E and scattering angle θ in the above Rutherford formular, refer to motion as viewed in the <u>center of mass frame of reference</u> - ie the frame of reference in which $\dot{\vec{R}} = 0$, the CM is stationary. The true total energy is

$$E^{tot} = \tfrac{1}{2} M \dot{\vec{R}}^2 + \tfrac{1}{2} \mu \dot{\vec{r}}^2 + U(r).$$ In the above, E refers only to $E = \tfrac{1}{2} \mu \dot{\vec{r}}^2 + U(r)$, the energy in the center of mass frame where $\dot{\vec{R}} = 0$. $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass of the projectile - target system.

In real experiments, one usually works in a "laboratory" frame where the target is stationary and the projectile is moving. One therefore needs to convert the energy E_{lab} of the projectiles and scattering angle θ_{lab} as measured in the lab frame, to the E_{cm} and θ_{cm} energy and angle as measured in the center of mass frame, before applying the Rutherford ~~next~~ formular.

$$b = \frac{k}{2E} \cot\left(\frac{\theta}{2}\right) = \frac{k}{2E} \frac{\cos(\theta/2)}{\sin(\theta/2)}$$

b ~ 1/θ

π   θ
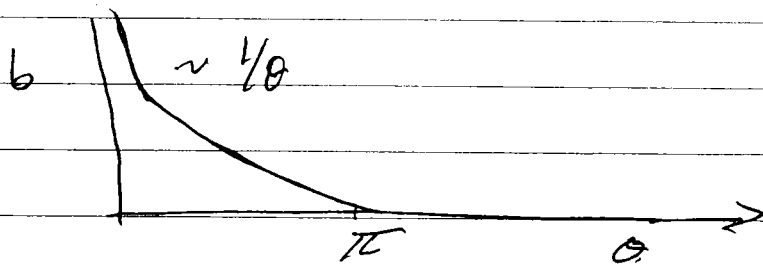
small θ gives large b

small b gives θ ≈ π

(back scattering

Rutherford scattering played a crucial role in early investigations of the structure of the atom

In the early 1900's there were two competing models for the atom: the Rutherford model of electrons orbiting a small positive nucleus, and the Thomson model of electrons embedded in a uniform sphere of positive charge the size of the atom. The key question: was the positive charge of the atom distributed over the entire size of the atom, a length scale of order $a_0 \sim .5 \text{Å}$, or was it concentrated on a much smaller length scale in a smaller nucleus?

Rutherford derived his scattering formula in 1911 and realized it could be used to answer this question. Since the electric field inside a uniformly charged sphere is <u>not</u>

Coulomb-like, but rather varies as $\bar{E} \sim \bar{r}$, there should be deviations from the Rutherford formula at large angles, corresponding to impact parameters b comparable in size to the radius over which the positive charge is distributed.

Observation of the angle beyond which the Rutherford formula breaks down then immediately determines the radius of the positive charge.

In 1913 Gieger + Morsden performed this experiment scattering 8 MeV α-particles (an α particle is a He nucleus) off of atoms in a gold foil. ~~Then~~ The dominant Coulomb interaction here is between the $+\cancel{2}e$ charge of the α-particle, and the positive nucleus of the Gold atoms — the gold electrons being uniformly distributed throughout the foil. By analyzing the differential crossssection they showed that the Rutherford formula held down to impact parameters of size $b \sim 10^{-14}$ m, which is much smaller than the 10^{-10} m size of the atom. This led to the Bohr model of the atom, and also showed that the Coulomb force obeys the $1/r^2$ behavior down to very small distances.

It is fortunate that the classically computed Rutherford cross section $d\sigma/d\Omega$ turns out to be identical to what one computes using quantum mechanic