Dynamics of many particle systems

In last section we considered the two-body problem. Now we consider when there are more than two particles. It is again convenient to divide our description of the motion of the system into the motion of the center of mass, and the relative motion of the particles about the center of mass

Center of mass

For particles m_i at positions $\vec{r}_i$, the center of mass position is

$$\vec{R} = \frac{1}{M} \sum_i m_i \vec{r}_i \quad \text{where} \quad M = \sum_i m_i \text{ is total mass}$$

For a solid body with a continuous mass distribution ρ

$$M = \int d^3r\, \rho \qquad d^3r = dx\,dy\,dz = dV$$

differential volume element

$$\vec{R} = \frac{1}{M} \int d^3r\, \rho\, \vec{r}$$

Linear momentum

Assume that the forces that act on the particles i are an external force $\vec{F}_i^{ex}$ plus internal forces from the other particles j, $\vec{f}_{ij}$. By Newton's 3rd law, $\vec{f}_{ij} = -\vec{f}_{ji}$, force on i due to j is equal and opposite to force on j due to i.

Total linear momentum is

$$\vec{P} = \sum_i \vec{p}_i = \sum_i m_i \frac{d\vec{r}_i}{dt} = \frac{d}{dt}\left(\sum_i m_i \vec{r}_i\right) = M\frac{d\vec{R}}{dt}$$

$$\boxed{\vec{P} = M\dot{\vec{R}}}$$

same as if all mass was located at the center of mass position $\vec{R}$.

$$\frac{d\vec{p}_i}{dt} = \vec{F}_i^{\,tot} = \vec{F}_i^{\,ex} + \sum_j \vec{f}_{ij}$$

$$\frac{d\vec{P}}{dt} = \sum_i \frac{d\vec{p}_i}{dt} = \sum_i \vec{F}_i^{\,ex} + \underbrace{\sum_i \sum_j \vec{f}_{ij}}$$

these terms all vanish in pairs since $\vec{f}_{ij} = -\vec{f}_{ji}$

$$\boxed{\dot{\vec{P}} = \sum_i \vec{F}_i^{\,ex}}$$

If total external force vanishes, then $\dot{\vec{P}} = 0$, or $\vec{P} =$ const, total linear momentum conserved.

$\Rightarrow$ if only forces are internal pairwise forces between the particles, total momentum $\vec{P}$ is conserved.

Angular Momentum

Let $\vec{r}_i = \vec{R} + \vec{r}_i\,'$, $\vec{r}_i\,'$ is position relative to center of mass

$$\vec{L} = \sum_i \vec{L}_i = \sum_i \vec{r}_i \times \vec{p}_i$$

$$= \sum_i (\vec{R} + \vec{r}_i\,') \times \vec{p}_i \qquad \text{use } \vec{p}_i = m_i \dot{\vec{r}}_i = m_i(\dot{\vec{R}} + \dot{\vec{r}}_i\,')$$

$$= \sum_i (\vec{R} + \vec{r}_i\,') \times m_i (\dot{\vec{R}} + \dot{\vec{r}}_i\,')$$

$$= \sum_i m_i \Big[\underset{①}{\vec{R} \times \dot{\vec{R}}} + \underset{②}{\vec{r}_i\,' \times \dot{\vec{R}}} + \underset{③}{\vec{R} \times \dot{\vec{r}}_i\,'} + \underset{④}{\vec{r}_i\,' \times \dot{\vec{r}}_i\,'} \Big]$$

term ② $= \sum_i m_i \vec{r}_i\,' \times \dot{\vec{R}} = \left(\sum_i m_i \vec{r}_i\,'\right) \times \dot{\vec{R}}$

term ③ $= \sum_i m_i \vec{R} \times \dot{\vec{r}}_i\,' = \vec{R} \times \frac{d}{dt}\left(\sum_i m_i \vec{r}_i\,'\right)$

both these terms vanish since by def of center of mass $\vec{R}$

$$\sum_i m_i \vec{r}_i\,' = \sum_i m_i (\vec{r}_i - \vec{R}) = \sum_i m_i \vec{r}_i - M\vec{R}$$
$$= M\vec{R} - M\vec{R} = 0.$$

term ① $= \sum_i m_i \vec{R} \times \dot{\vec{R}} = \vec{R} \times M\dot{\vec{R}} = \vec{R} \times \vec{P}$

total momentum

term ④ $= \sum_i m_i \vec{r}_i\,' \times \dot{\vec{r}}_i\,' = \sum_i \vec{r}_i\,' \times \vec{p}_i\,'$ where $\vec{p}_i\,' = m\dot{\vec{r}}_i\,'$ is momentum with respect to center of mass

$$\Rightarrow \boxed{\vec{L} = \vec{R} \times \vec{P} + \sum_i \vec{r}_i\,' \times \vec{p}_i\,'}$$

= angular momentum of center of mass about the origin
+ angular momentum of particles about the center of mass

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\left(\sum_i \vec{r}_i \times \vec{p}_i\right) = \sum_i \left(\dot{\vec{r}}_i \times \vec{p}_i + \vec{r}_i \times \dot{\vec{p}}_i\right)$$

but $\dot{\vec{r}}_i \times \vec{p}_i = \dot{\vec{r}}_i \times m_i \dot{\vec{r}}_i = 0$ as $\dot{\vec{r}}_i \times \dot{\vec{r}}_i = 0$

So
$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \dot{\vec{p}}_i = \sum_i \vec{r}_i \times \vec{F}_i^{\,tot}$$

$$= \sum_i \vec{r}_i \times \left(\vec{F}_i^{\,ex} + \sum_j \vec{f}_{ij}\right)$$

$$= \sum_i \vec{r}_i \times \vec{F}_i^{\,ex} + \sum_{i,j} \vec{r}_i \times \vec{f}_{ij}$$

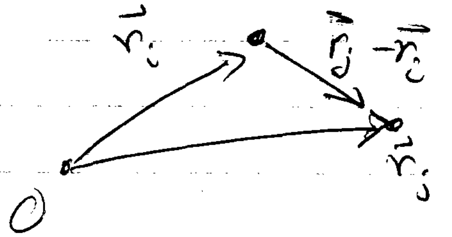
Consider the last term:

$$\sum_{ij} \vec{r}_i \times \vec{f}_{ij} = \sum_{ji} \vec{r}_j \times \vec{f}_{ij}$$ changing the names of the dummy indices

$$= -\sum_{ji} \vec{r}_j \times \vec{f}_{ij}$$ as $\vec{f}_{ij} = -\vec{f}_{ji}$

So $\sum_{ij} \vec{r}_i \times \vec{f}_{ij} = \frac{1}{2} \sum_{ij} \left(\vec{r}_i \times \vec{f}_{ij} - \vec{r}_j \times \vec{f}_{ij}\right) = \sum_{ij} (\vec{r}_i - \vec{r}_j) \times \vec{f}_{ij}$

For *central* forces, $\vec{f}_{ij}$ is in direction pointing from particle i to particle j. This is the same direction as the vector $\vec{r}_j - \vec{r}_i = -(\vec{r}_i - \vec{r}_j)$



since $\vec{r}_i - \vec{r}_j$ and $\vec{f}_{ij}$ are parallel vectors, $(\vec{r}_i - \vec{r}_j) \times \vec{f}_{ij} = 0$

$\Rightarrow$ for central forces

$$\boxed{\dot{\vec{L}} = \sum_i \vec{r}_i \times \vec{F}_i^{ex} = \sum_i \vec{N}_i^{ex}}$$

where $\vec{N}_i^{ex} = \vec{r}_i \times \vec{F}_i^{ex}$ is external torque on particle i

if the total external torque is zero, then $\dot{\vec{L}} = 0$, $\vec{L} = \text{const}$, angular momentum is conserved.

Energy of the System

$$\dot{\vec{r}}_i = \dot{\vec{R}} + \dot{\vec{r}}_i{}'$$

kinetic energy $T = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i{}^2$

$$= \frac{1}{2} \sum_i m_i \left(\dot{\vec{R}} + \dot{\vec{r}}_i{}'\right) \cdot \left(\dot{\vec{R}} + \dot{\vec{r}}_i{}'\right)$$

$$= \frac{1}{2} \sum_i \left[m_i (\dot{\vec{R}})^2 + m_i (\dot{\vec{r}}_i{}')^2 + 2 m_i \dot{\vec{R}} \cdot \dot{\vec{r}}_i{}' \right]$$

$$= \frac{1}{2} M V^2 + \sum_i \frac{1}{2} m_i v_i'^2 + \dot{\vec{R}} \cdot \frac{d}{dt} \underbrace{\left(\sum_i m_i \vec{r}_i{}' \right)}_{= 0}$$

as before

$$\boxed{T = \frac{1}{2} M V^2 + \sum_i \frac{1}{2} m_i v_i'^2}$$

total kinetic energy is kinetic energy of center of mass motion plus kinetic energy of motion with respect to the center of mass

From your earlier mechanics course you know that if a particle is moved from position "1" to position "2", the change in kinetic energy is equal to the work done on the particle by the forces that act on it,

$$T_i(2) - T_i(1) = \int_1^2 d\vec{r}_i \cdot \vec{F}_i$$

For a system of many particles, the change in total kinetic energy is

$$T_2 - T_1 = \sum_i \int_1^2 d\vec{r}_i \cdot \vec{F}_i = \sum_i \int_1^2 d\vec{r}_i \cdot \left(\vec{F}_i^{ex} + \sum_j \vec{f}_{ij} \right)$$

define $f_{ii} \equiv 0$

Assume now that all forces are conservative, ie

$\vec{F}_i^{ex} = -\vec{\nabla}_i U_i$ where U_i is the external potential energy for particle i, and $\vec{\nabla}_i$ means spatial derivatives with respect to $\vec{r}_i$

cf. $F_{ix} = -\dfrac{\partial U_i(\vec{r}_i)}{\partial x_i}$

$\vec{f}_{ij} = -\vec{\nabla}_i \tilde{U}_{ij}$ where $\tilde{U}_{ij}$ is the potential energy for the force between particles i and j

$\tilde{U}_{ij}$ depends only on $\vec{r}_i - \vec{r}_j$, $\tilde{U}_{ij}(\vec{r}_i - \vec{r}_j) \equiv \tilde{U}_{ij}(\vec{r}_{ij})$

and $-\vec{\nabla}_i \tilde{U}_{ij} = -\dfrac{\partial}{\partial \vec{r}_i} \tilde{U}_{ij}(\vec{r}_i - \vec{r}_j)$

by Newtons 3rd Law, $\vec{f}_{ij} = -\vec{f}_{ji}$, this follows from above since,

$$\Rightarrow \frac{\partial}{\partial \vec{r}_i} \tilde{U}_{ij}(\vec{r}_i - \vec{r}_j) = -\frac{\partial}{\partial \vec{r}_j} \tilde{U}_{ji}(\vec{r}_i - \vec{r}_j)$$

We want to rewrite the work - kinetic energy theorem above as a conservation of energy law.

Now $\sum_i \int_1^2 d\vec{r}_i \cdot \vec{F}_i^{ext} = -\sum_i \int_1^2 d\vec{r}_i \cdot \vec{\nabla}_i U_i = -\sum_i \left[U_i(2) - U_i(1)\right]$

$$\sum_i \int d\vec{r}_i \cdot \sum_j \vec{f}_{ij} = \sum_{i,j} \int d\vec{r}_i \cdot \vec{f}_{ij} = \sum_{j,i} \int d\vec{r}_j \cdot \vec{f}_{ji}$$

$= -\sum_{j,i} \int d\vec{r}_j \cdot \vec{f}_{ij}$ as $\vec{f}_{ij} = -\vec{f}_{ji}$ switch dummy indices of summation

$$\Rightarrow \sum_i \int d\vec{r}_i \cdot \sum_j \vec{f}_{ij} = \frac{1}{2} \sum_{i,j} \int \vec{f}_{ij} \cdot (d\vec{r}_i - d\vec{r}_j)$$

$= \frac{1}{2} \sum_{ij} \int_1^2 f_{ij} \cdot d\vec{r}_{ij}$ $\qquad d\vec{r}_{ij} \equiv d\vec{r}_i - d\vec{r}_j$

Now for $\tilde{U}_{ij}(\vec{r}_{ij})$, $f_{ij} = -\frac{\partial}{\partial \vec{r}_i} \tilde{U}_{ij}(\vec{r}_i - \vec{r}_j)$

$$= -\frac{\partial}{\partial \vec{r}_{ij}} \tilde{U}_{ij}(\vec{r}_{ij})$$

so

$$= -\frac{1}{2} \sum_{ij} \left[\tilde{U}_{ij}(2) - \tilde{U}_{ij}(1)\right]$$

So combining we get

$$T_2 - T_1 = -\sum_i \left(U_i(2) - U_i(1)\right) - \frac{1}{2} \sum_{ij} \left(\tilde{U}_{ij}(2) - \tilde{U}_{ij}(1)\right)$$

$$\Rightarrow T_2 + \sum_i U_i(2) + \frac{1}{2} \sum_{ij} \tilde{U}_{ij}(2) = T_1 + \sum_i U_i(1) + \frac{1}{2} \sum_{ij} \tilde{U}_{ij}(1)$$

If define ~~total~~ potential energy

$$U = \underbrace{\sum_i U_i}_{\text{external pot energy}} + \underbrace{\frac{1}{2} \sum_{ij} U_{ij}}_{\text{internal pot energy}}$$

Then we have

$$T_2 + U_2 = T_1 + U_1$$

or

defining total mechanical energy $E = T + U$

$$E_2 = E_1$$ energy is conserved.
if all forces are conservative.