Non inertial frames of reference

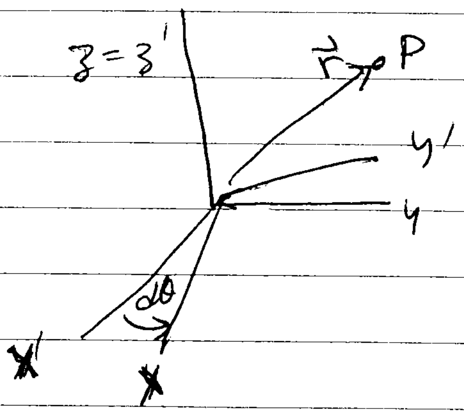
Newtons laws, in particular $\vec{F} = m\vec{a}$, only hold in inertial frames of reference.

How can we describe motion in a frame of reference that is accelerating with respect to an inertial frame? We will see that in such a frame of reference, the acceleration of the frame leads to "fictitious" forces, ie Newton's 2nd law will have the form

$$m\vec{a} = \vec{F} + \vec{F}^{fic}$$

where $\vec{a}$ is the accel in the non-inertial frame, $\vec{F}$ are the true physical forces that act on the mass, and $\vec{F}^{fic}$ are addition terms related to the acceleration of the non-inertial frame with respect to the inertial one.

Consider two sets of coordinate frames



inertial frame: "fixed": $x', y', z' \rightarrow \vec{r}\,'$
non inertial frame: "rotating": x, y, z
$\rightarrow \vec{r}$

Consider the point P at position $\vec{r}$ as measured in the rotating frame.

If point P is stationary in the rotating frame, then after an infinitesimal rotation, we have from chpt 1, that the position in the fixed frame will be

$$\vec{r}\,' = \vec{r} + d\vec{\theta} \times \vec{r}$$

In infinitesimal time dt, the rate of change in position will be

$$\frac{d\vec{r}\,'}{dt} = \frac{d\vec{r}}{dt} + \frac{d\vec{\theta}}{dt} \times \vec{r} = \frac{d\vec{r}}{dt} + \vec{\omega} \times \vec{r}$$

If P is stationary in the rotating frame $\frac{d\vec{r}}{dt} = 0$ and $\frac{d\vec{r}\,'}{dt} = \vec{\omega} \times \vec{r}$, as we had in Chpt 1.

If now P is moving with respect to the rotating frame, we get

$$\frac{d\vec{r}\,'}{dt} = \frac{d\vec{r}}{dt} + \vec{\omega} \times \vec{r}$$

or in notation of the text

$$\left(\frac{d\vec{r}}{dt}\right)_{fixed} = \left(\frac{d\vec{r}}{dt}\right)_{rotating} + \vec{\omega}\times\vec{r}$$

$\vec{r}$ here is position in rotating frame

Similarly for any vector quantity $\vec{Q}$ which can be measured in either the fixed or the rotating frame, we have

$$\left(\frac{d\vec{Q}}{dt}\right)_{fixed} = \left(\frac{d\vec{Q}}{dt}\right)_{rotating} + \vec{\omega}\times\vec{Q}$$

Apply above to the angular velocity vector $\vec{\omega}$

$$\left(\frac{d\vec{\omega}}{dt}\right)_{fixed} = \left(\frac{d\vec{\omega}}{dt}\right)_{rotating} + \underbrace{\vec{\omega}\times\vec{\omega}}_{=0}$$

$$\left(\frac{d\vec{\omega}}{dt}\right)_{fixed} = \left(\frac{d\vec{\omega}}{dt}\right)_{rotating}$$

$\Rightarrow$ angular acceleration $\dot{\vec{\omega}}$ is the same in both fixed and rotating frames

Apply above to the basis vectors $\hat{e}_i$ of the rotating frame

$$\left(\frac{d\hat{e}_i}{dt}\right)_{fixed} = \left(\frac{d\hat{e}_i}{dt}\right)_{rotating} + \vec{\omega}\times\hat{e}_i$$

but in the rotating frame, $\hat{e}_i$ are stationary

$$\Rightarrow \left(\frac{d\hat{e}_i}{dt}\right)_{rot} = 0$$

$$\Rightarrow \left(\frac{d\hat{e}_i}{dt}\right)_{fixed} = \mathring{\hat{e}}_i = \omega \times \hat{\mathbf{e}}_i$$

we used this to get $\mathring{\hat{e}}_r = \dot{\theta}\,\hat{e}_\theta$

$\mathring{\hat{e}}_\theta = -\dot{\theta}\,\hat{e}_r$

in polar coords.

Suppose now that the origins of the rotating & the fixed coord systems do not coincide, but are separated by a distance $\vec{R}$. If $\vec{R}$ varies with time we get $\quad \vec{r}\,' = \vec{R} + \vec{r} + d\vec{\theta} \times \vec{r}$

$$\Rightarrow \left(\frac{d\vec{r}\,'}{dt}\right)_{fixed} = \left(\frac{d\vec{R}}{dt}\right)_{fixed} + \left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega} \times \vec{r}$$

define: $\vec{v}_{fix} = \mathring{\vec{r}}_{fix} \equiv \left(\frac{d\vec{r}\,'}{dt}\right)_{fixed}$ velocity of particle in fixed frame

$\vec{V} = \mathring{\vec{R}}_{fix} \equiv \left(\frac{d\vec{R}}{dt}\right)_{fixed}$ velocity of rotating frame as measured in fixed frame

$\vec{v}_{rot} = \mathring{\vec{r}}_{rot} \equiv \left(\frac{d\vec{r}}{dt}\right)_{rot}$ velocity of particle measured with respect to rotating frame.

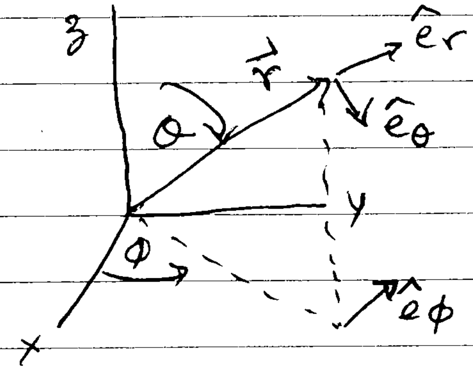
then

$$\vec{v}_{fix} = \vec{V} + \vec{v}_{rot} + \vec{\omega} \times \vec{r}$$

Exaple

Spherical coordinates

New method to get old results!



when particle moves from $\vec{r}$ to $\vec{r}+d\vec{r}$, the radial coord changes by dr, and the direction changes according to the infinitesimal rotation

$$d\theta\, \hat{e}_\phi + d\phi\, \hat{e}_z$$

This gives an instantaneous angular velocity of

$$\vec{\omega} = \dot{\theta}\, \hat{e}_\phi + \dot{\phi}\, \hat{e}_z$$

$$\dot{\hat{e}}_r = \vec{\omega} \times \hat{e}_r = \dot{\theta}\,(\hat{e}_\phi \times \hat{e}_r) + \dot{\phi}\,(\hat{e}_z \times \hat{e}_r)$$

$$\dot{\hat{e}}_r = \dot{\theta}\, \hat{e}_\theta + \dot{\phi} \sin\theta\, \hat{e}_\phi$$

$$\dot{\hat{e}}_\theta = \vec{\omega} \times \hat{e}_\theta = \dot{\theta}\,(\hat{e}_\phi \times \hat{e}_\theta) + \dot{\phi}\,(\hat{z} \times \hat{e}_\theta)$$

$$= -\dot{\theta}\, \hat{e}_r + \sin\left(\theta + \frac{\pi}{2}\right) \dot{\phi}\, \hat{e}_\phi$$

$$\dot{\hat{e}}_\theta = -\dot{\theta}\, \hat{e}_r + \dot{\phi} \cos\theta\, \hat{e}_\phi$$

$$\dot{\hat{e}}_\phi = \vec{\omega} \times \hat{e}_\phi = \dot{\theta}\,(\hat{e}_\phi \times \hat{e}_\phi) + \dot{\phi}\,(\hat{e}_z \times \hat{e}_\phi)$$

$$= 0 + \dot{\phi}\left[\cos\theta\, \hat{e}_r - \sin\theta\, \hat{e}_\theta\right] \times \hat{e}_\phi$$

$$= \dot{\phi}\left[\cos\theta\,(\hat{e}_r \times \hat{e}_\phi) - \sin\theta\,(\hat{e}_\theta \times \hat{e}_\phi)\right]$$

$$= -\dot{\phi} \cos\theta\, \hat{e}_\theta - \dot{\phi} \sin\theta\, \hat{e}_r$$

$$\dot{\hat{e}}_\phi = -\dot{\phi}\,(\sin\theta\, \hat{e}_r + \cos\theta\, \hat{e}_\theta)$$

Same result for time rate of change of spherical basis vectors as found earlier at start of semester velocity in spherical coords

$$\vec{v}_{fix} = \vec{V} + \vec{v}_{rot} + \vec{\omega} \times \vec{r}$$

Here $\vec{v}_{fix}$ is the velocity in the fixed frame – this is what we want, we earlier called it just $\vec{v}$

$\vec{V} = 0$ as origin of rotating frame is stationary

$\vec{v}_{rot} = \dot{r}\,\hat{e}_r$ as seen in the rotating frame

$\vec{\omega} = \dot{\theta}\,\hat{e}_\phi + \dot{\phi}\,\hat{e}_z$

$$\Rightarrow \vec{v} = \dot{r}\,\hat{e}_r + (\dot{\theta}\,\hat{e}_\phi + \dot{\phi}\,\hat{e}_z) \times r\,\hat{e}_r$$

$$= \dot{r}\,\hat{e}_r + r\dot{\theta}\,(\hat{e}_\phi \times \hat{e}_r) + r\dot{\phi}\,(\hat{e}_z \times \hat{e}_r)$$

$$\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta + r\dot{\phi}\sin\theta\,\hat{e}_\phi$$

same result as found earlier

Centrifugal + Coriolis forces

$$\vec{F} = m\vec{a}_{fix} = m\left(\frac{d\vec{v}_{fix}}{dt}\right)_{fixed}$$

$$\left(\frac{d\vec{v}_{fix}}{dt}\right)_{fixed} = \left(\frac{d\vec{V}}{dt}\right)_{fixed} + \left(\frac{d\vec{v}_{rot}}{dt}\right)_{fixed} + \dot{\vec{\omega}}\times\vec{r} + \omega\times\left(\frac{d\vec{r}}{dt}\right)_{fixed}$$

$$\dot{\vec{\omega}}_{fixed} = \dot{\vec{\omega}}_{rot}$$

call $\left(\frac{d\vec{V}}{dt}\right)_{fixed} = \ddot{\vec{R}}_{fix}$

$$\left(\frac{d\vec{v}_{rot}}{dt}\right)_{fixed} = \left(\frac{d\vec{v}_{rot}}{dt}\right)_{rotating} + \vec{\omega}\times\vec{v}_{rot}$$

$$= \vec{a}_{rot} + \vec{\omega}\times\vec{v}_{rot}$$

acceleration with respect to rotating coord

$$\vec{a}_{rot} = \ddot{\vec{r}}_{rot}$$

$$\omega\times\left(\frac{d\vec{r}}{dt}\right)_{fixed} = \vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$= \vec{\omega}\times\vec{v}_{rot} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

Combining

~~$\vec{F}_{effective} = m\vec{a}_{rot}$~~

$$\vec{F} = m\vec{a}_{fix} = m\ddot{\vec{R}}_{fix} + m\vec{a}_{rot} + m\vec{\omega}\times\vec{v}_{rot} + m\dot{\vec{\omega}}\times\vec{r} + m\vec{\omega}\times\vec{v}_{rot} + m\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

transl accel of rotating frame ↓ angular accel of rotating frame ↓

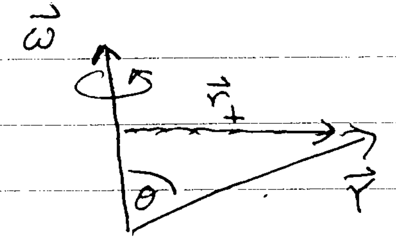
$$\text{or} \quad \vec{F}_{effective} \equiv m\vec{a}_{rot} = \vec{F} - m\ddot{\vec{R}}_{fix} - m\dot{\vec{\omega}} \times \vec{r}$$
$$- m\,\vec{\omega} \times (\vec{\omega} \times \vec{r})$$
$$- 2m\,\vec{\omega} \times \vec{v}_{rot}$$

all the terms besides $\vec{F}$ on the right hand side are "fictitious" forces – they lead to acceleration $\vec{a}_{rot}$ as viewed in the rotating frame – they do not lead to any real acceleration $\vec{a}_{fix}$ as seen in the fixed frame.

the terms $-m\ddot{\vec{R}}_{fix}$ and $-m\dot{\vec{\omega}} \times \vec{r}$ result from the translational and rotational acceleration of the rotating frame.

the term $-m\vec{\omega} \times (\vec{\omega} \times \vec{r})$ is called the centrifugal force. For rotation about the z axis and r in the xy plane, for example, $-m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = m\omega^2\vec{r}$ points in outward radial direction.

the term $-2m\vec{\omega} \times \vec{v}_{rot}$ is called the Coriolis force exists only when particle is moving with respect to the rotating frame, ie $\vec{v}_{rot} \neq 0$.



$|\vec{r}_\perp| = r\sin\theta$

$-m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = m\omega^2\vec{r}_\perp$

centrifugal force

If we view $\vec{\omega}$ as along $\hat{z}$ axis, then centrifugal force is in outward acceleration

$$\vec{a}_{fix} = \ddot{\vec{R}}_{fix} + \vec{a}_{rot} + 2\vec{\omega}\times\vec{v}_{rot} + \dot{\vec{\omega}}\times\vec{r}$$
$$+ \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$\vec{a}_{fix} \equiv \vec{a}$$

$$\ddot{\vec{R}}_{fix} = 0 \qquad \vec{v}_{rot} = \dot{r}\,\hat{e}_r$$

$$\vec{a}_{rot} = \ddot{r}\,\hat{e}_r$$

$$\vec{\omega} = \dot{\theta}\,\hat{e}_\phi + \dot{\phi}\,\hat{e}_z = \dot{\theta}\,\hat{e}_\phi + \dot{\phi}\cos\theta\,\hat{e}_r - \dot{\phi}\sin\theta\,\hat{e}_\theta$$

$$\dot{\vec{\omega}} = \ddot{\theta}\,\hat{e}_\phi + \dot{\theta}\,\dot{\hat{e}}_\phi + \ddot{\phi}\,\hat{e}_z$$
$$= \ddot{\theta}\,\hat{e}_\phi - \dot{\theta}\dot{\phi}(\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta) + \ddot{\phi}(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta)$$

$$\dot{\vec{\omega}} = (\ddot{\phi}\cos\theta - \dot{\theta}\dot{\phi}\sin\theta)\,\hat{e}_r - (\ddot{\phi}\sin\theta + \dot{\theta}\dot{\phi}\cos\theta)\,\hat{e}_\theta$$
$$+ \ddot{\theta}\,\hat{e}_\phi$$

$$\dot{\vec{\omega}}\times\vec{r} = \dot{\vec{\omega}}\times r\hat{e}_r = -(\ddot{\phi}\sin\theta + \dot{\theta}\dot{\phi}\cos\theta)\,r\,\hat{e}_\theta\times\hat{e}_r$$
$$+ r\ddot{\theta}\,\hat{e}_\phi\times\hat{e}_r$$

$$\dot{\vec{\omega}}\times\vec{r} = (r\ddot{\phi}\sin\theta + r\dot{\theta}\dot{\phi}\cos\theta)\,\hat{e}_\phi + r\ddot{\theta}\,\hat{e}_\theta$$

$$\vec{\omega}\times\vec{v}_{rot} = (\dot{\theta}\,\hat{e}_\phi + \dot{\phi}\cos\theta\,\hat{e}_r - \dot{\phi}\sin\theta\,\hat{e}_\theta)\times\dot{r}\,\hat{e}_r$$

$$= \dot{r}\dot{\theta}\,\hat{e}_\theta + \dot{r}\dot{\phi}\sin\theta\,\hat{e}_\phi$$

$$\vec{\omega} \times \vec{r} = (\dot{\theta}\,\hat{e}_\phi + \dot{\phi}\cos\theta\,\hat{e}_r - \dot{\phi}\sin\theta\,\hat{e}_\theta) \times r\hat{e}_r$$

$$= r\dot{\theta}\,\hat{e}_\theta + r\dot{\phi}\sin\theta\,\hat{e}_\phi$$

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = (\dot{\theta}\,\hat{e}_\phi + \dot{\phi}\cos\theta\,\hat{e}_r - \dot{\phi}\sin\theta\,\hat{e}_\theta) \times (r\dot{\theta}\,\hat{e}_\theta + r\dot{\phi}\sin\theta\,\hat{e}_\phi$$

$$= -r\dot{\theta}^2\,\hat{e}_r + r\dot{\theta}\dot{\phi}\cos\theta\,\hat{e}_\phi$$

$$- r\dot{\phi}^2\sin\theta\cos\theta\,\hat{e}_\theta - r\dot{\phi}^2\sin^2\theta\,\hat{e}_r$$

$$\vec{a}_{fix} = \ddot{r}\,\hat{e}_r + 2\left(\dot{r}\dot{\theta}\,\hat{e}_\theta + \dot{r}\dot{\phi}\sin\theta\,\hat{e}_\phi\right)$$
$$+ \left(r\ddot{\phi}\sin\theta + r\dot{\theta}\dot{\phi}\cos\theta\right)\hat{e}_\phi + r\ddot{\theta}\,\hat{e}_\theta$$
$$+ - r\dot{\phi}^2\sin\theta\cos\theta\,\hat{e}_\theta - r\dot{\phi}^2\sin^2\theta\,\hat{e}_r$$

$$\vec{a}_{fix} = \left(\ddot{r} - r\dot{\phi}^2\sin^2\theta\right)\hat{e}_r$$

$$+ \left(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta\right)\hat{e}_\theta$$

$$+ \left(r\ddot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + \dot{r}\dot{\phi}\sin\theta\right)\hat{e}_\phi$$

If ~~no acceleration, ie~~ $\cancel{\ddot{r} = \ddot{\theta} = \ddot{\phi} = 0}$ ~~then~~

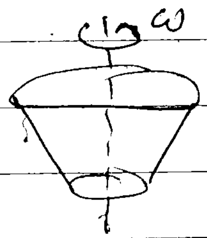
$$\cancel{\vec{a} = -r\dot{\phi}^2\sin^2\theta\,\hat{e}_r + \left(2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta\right)\hat{e}_\theta}$$
$$\cancel{+ \left(r\dot{\theta}\dot{\phi}\cos\theta + \dot{r}\dot{\phi}\sin\theta\right)\hat{e}_\phi}$$

~~If~~ $\cancel{\dot{\theta} = 0}$

$$\cancel{\vec{a} = \left(\ddot{r} - r\dot{\phi}^2\sin^2\theta\right)\hat{e}_r - r\dot{\phi}^2\sin\theta\cos\theta\,\hat{e}_\theta}$$
$$\cancel{+ \left(r\ddot{\phi}\sin\theta + \dot{r}\dot{\phi}\sin\theta\right)e_\phi}$$

Example - Surface of liquid in a rotating bucket

A ~~one~~ bucket of water is rotating with angular speed ω about an axis of rotational symmetry. Find the shape of the surface of the water.



Assume that the liquid is at rest in the rotating frame of the bucket. The only one of the "fictitious" forces that is not zero is the centrifugal force.

The total external force on an element of mass m of the water surface is the pressure gradient plus gravity

$$\vec{F} = \vec{F}_{pressure} + m\vec{g}$$

where $\vec{F}_{pressure}$ is always normal to the surface, and $\vec{g} = -g\hat{z}$ is downward

In the rotating frame, the total effective force is

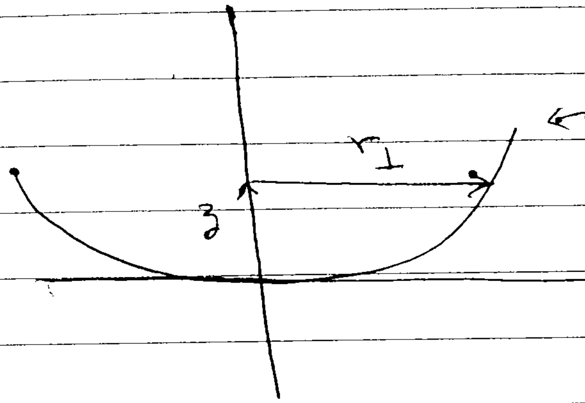
$$\vec{F}^{*}_{rot} = \vec{F} - m\,\vec{\omega}\times(\omega\times\vec{r}) = \vec{F}_{pressure} + m\,\vec{g}_{eff}$$

where $\vec{g}_{eff} = \vec{g} - \vec{\omega}\times(\vec{\omega}\times\vec{r})$

Since the water is not moving in the rotating frame,
$\vec{a}_{rot} = 0 \Rightarrow \vec{F}_{rot} = 0 \Rightarrow \vec{F}_{pressure} = -m\,\vec{g}_{eff}$

Since $\vec{F}_{pressure}$ is normal to the surface, so must $\vec{g}_{eff}$ be centrif force

$$\vec{g}_{eff} = -g\,\hat{z} + \omega^2 r_\perp\,\hat{r}_\perp$$



← surface of water is $z(r_\perp)$

If $\vec{g}_{eff}$ is normal to the surface then

$$(dr_\perp\,\hat{r}_\perp + dz\,\hat{z})\cdot\vec{g}_{eff} = 0$$

$$\Rightarrow dr_\perp\,\vec{g}_{eff}\cdot\hat{r}_\perp + dz\,\vec{g}_{eff}\cdot\hat{z}$$

$$= dr_\perp(\omega^2 r_\perp) + dz(-g) = 0$$

$$\Rightarrow \frac{dz}{dr_\perp} = \frac{\omega^2 r_\perp}{g} \Rightarrow \boxed{z(r_\perp) = \frac{1}{2}\frac{\omega^2 r_\perp^2}{g} + \text{const}}$$

Surface is a paraboloid.

Make parabolic mirrors for telescopes by spinning a large vat of molten glass while it solidifies